THE TEACHING OF JUNIOR HIGH SCHOOL MATHEMATICS

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GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

PRINTED IN THE UNITED STATES OF AMERICA

627.7

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

The gratifying development of the junior high school in this country has given rise in recent years to two important lines of investigation. The first concerns itself with the problems of the curriculum, and the second with the improvement of the quality of instruction. Each has raised a series of questions in the minds of teachers and administrators, and upon the answers that shall be given depends in a large degree the success or failure of this new type of school. Inspired to a considerable extent by the work of the International Commission on the Teaching of Mathematics, by the report of the National Committee appointed by the Mathematical Association of America, and by the results of the testing movement, teachers have been seeking these answers and have, through local, state, and national organizations, seriously studied the problems that have arisen. The result has been a body of conclusions that deserve the attention of the schools and of the public at large.

In this work the authors have set forth the leading problems that have been considered, the conclusions generally reached, the questions needing further study, and some of the means that tend to make this study more effective. They have also made various suggestions for improving the curriculum, the methods of instruction, and the testing of pupils; for selecting the objectives in each of the important branches of mathematics in the junior high school; for making the subject seem more real and interesting; and for the treatment of various other topics as set forth in the table of contents.

If properly organized and carefully presented, no subject in the curriculum should ever seem dull or uninteresting. Even if a retarded group is unable to keep pace with the rest of the class, the work should be so planned as to appeal to its members and lie within their ready grasp. The same is true with respect to the more brilliant pupils, — often the most retarded in op-

portunity to advance; their work should be interesting to their higher intelligence and should be worthy of their superior powers. It is a sad confession to make that our standard of teaching is too often set by the demands of mediocrity. It is one of the purposes of this book to show how the curriculum can be prepared to meet the needs of three general types of pupils, — the two extremes referred to above and the large middle group commonly known as the average. The administrative problem is by no means easy of solution in the smaller schools, and hence the results will naturally depend largely upon local conditions and upon the needs and abilities of each school system.

Our present problem is, however, something more than a mere listing of objectives, the stating of minimum essentials, or the differentiating of the material within our courses. It is largely a matter of developing better teachers, — teachers of improved and enlarged academic training in mathematics, who are versed in the psychology of childhood and are possessed of such a love for children as to make the work of teaching a delight. Given the most ideal curriculum and the most promising pupils possible, still with a "humdrum" teacher who lacks the elements mentioned the work is bound to be uninspiring and the results unsatisfactory. No book, however stimulating, can wholly remodel such a teacher or make up for inborn defects. On the other hand, for those who give real promise of success in teaching mathematics it is hoped that the message herein contained will be helpful.

Since the book will be used chiefly in institutions of collegiate grade and by those who are preparing for positions of more than usual responsibility, there have been included numerous bibliographies and lists of questions and topics for discussion. These will, it is believed, serve as useful aids to instruction. The text itself, however, has been prepared quite as much for the use of the individual teacher who has no opportunity to attend college classes as for those who, in increasing numbers, frequent our summer schools and university courses.

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CHAPTER I

HOW THE CURRICULUM IS DETERMINED

1. Defining the Objectives

Need for Objectives. In teaching mathematics, as in any other kind of work we set out to do, we cannot expect to realize our aims unless they are precise and are clearly defined at the outset. The first step, therefore, in planning a modern course of study in mathematics for the junior high school is to prepare a list of desirable objectives which we hope to attain. Such a list should not be imposed upon the schools by some higher authority; on the contrary, it should be the result of much thought and discussion on the part of those who will actually use it, who know the work of the classroom, and who themselves are the authority as to what can be expected of children. If properly made and carefully checked, a list of this kind is of value, not merely to the inexperienced teacher but to those who have already become accustomed to consider the constantly changing problems of the proper selection of material.

Nature of Objectives. There are two kinds of objectives to be kept in mind,—the great central ones and those that are more specifically mathematical. The former have to do with the realization of aims that are not peculiar to mathematics alone but which are to be sought in all fields of knowledge. Thus, the central objective in teaching junior-high-school mathematics is to develop well-educated citizens. This purpose is much the same in the teaching of all the great departments of knowledge, such as literature, history, language, art, mathematics, natural

science, religion, and contemporary civilization. Everyone has a right to know the general nature of each of these departments. Mathematics being one of them, a strictly mathematical objective is to give the pupils in the junior high school some idea of the general nature and uses of both business and social arithmetic, of intuitive geometry, of practical algebra, and of numerical trigonometry of a simple type, and some knowledge of the meaning of a demonstration in geometry. In other words, we should give as complete and all-round a view of the nature of mathematics as possible in the time allowed.

Function of the Junior High School. The junior high school should therefore offer an opportunity for making a general survey of the meaning of the great branches of human knowledge. It should not fear to be superficial, for its mission is to show a wide surface rather than a narrow depth. It should be at the same time a period in which interest should be awakened, knowledge extended, and a diagnosis made of the pupil's abilities and tastes. At its close each pupil should have shown his parents, his teachers, and himself, in a rather general way, what is his natural bent of mind and what it will very likely continue to be. If he has no taste for or ability with respect to mathematics, and does not expect to enter a college or a technical school, he should no longer be required to pursue the subject, and similarly with respect to other important branches. His mind has been given its chance; it should now be given a reasonable choice, — a matter to be determined by his parents, his teachers, and himself. It is not likely to be such as to lead to any narrow specialization in the senior high school, but it is quite possible that it will lead him to cease studying certain special subjects, — say natural science, foreign language, mathematics, drawing, or the manual arts, — for which he has no taste and in which he has shown no ability. This does not mean unrestrained license of election or freedom to choose "snap courses"; it simply means that there is little use in continuing in a field in which there is neither interest nor hope, or in attempting to proceed to college when the mind does not permit of laying the necessary foundations.

Classification of Mathematical Objectives. For convenience, mathematical objectives also may be classified in two groups, — the great central mathematical objectives and the more specific mathematical objectives, the latter being subsidiary in the entire scheme. These objectives may be listed in various ways; they may be placed in alphabetical sequence like the words in a dictionary, they may be arranged according to their difficulty for the pupils, they may be classified according to the branches of mathematics commonly taught, they may be set forth with little or no regard to pedagogical sequence, or they may be given in the order which experience has shown to be the most helpful to teachers, even though there is some sacrifice of logical sequence and some duplication of details. In this discussion, as well as in the more elaborate treatment in Chapter III, we shall follow the last of these five plans.

List not Final or Complete. As will later be stated in Chapter III, where the matter will be more fully discussed, the lists of objectives should not be considered as either final or complete; in fact, in this case, it is purposely left incomplete. To submit a longer list at present might be wearisome and discouraging and thus do more harm than good to teachers who are trying to clarify their views. On the other hand, too short a list might mean that the pupils throughout the country would not have the advantage of a proper survey of the possible field. The teacher who feels so inclined and has reasonable confidence in himself should modify the list to meet the needs of the community which he is to serve, keeping in mind that some things in the course should be uniformly taught everywhere. In making the selection emphasis should be placed upon the fact that everything included should be good material; in other words, the objectives should be fundamental.

Basis of Choosing Objectives. The objectives selected have been chosen after a careful consideration of three important criteria: (1) the intrinsic worth of mathematics itself; (2) the social needs of people in general, and especially those in the local community; and (3) the interests of the pupils who are to study the subject. The first criterion requires that the

teacher must know mathematics as thoroughly as possible in order to judge the value of its various branches. In any case, he owes it to himself and to his pupils to increase his knowledge continually. The second criterion means that the future well-being and social status of the nation and community must be considered with respect to what mathematics will be useful to the average, well-educated citizen. The third criterion means that the objectives must be such that the material chosen for each grade shall be that which is likely to be the most valuable to the pupil if he should leave school at the end of that year. Moreover, it is important that the pupil be led to see the real worth of the objectives chosen in order that the chance of realizing them may thus be increased.

2. Selection of Activities and Material

Choice of Subject Matter. In the past, various methods of choosing the subject matter of mathematics have been used. Although much good material has been selected by each of these methods, most if not all of them have their disadvantages. The method of selecting material from various courses of study in use throughout the country, although representative of what is being taught, is open to serious objection, it being well known that such courses generally tend to perpetuate obsolete processes and antiquated business methods, and usually fail to be of help in suggesting the thing which ought to be taught.

It is equally true that the best material cannot be secured by making an inventory of the current textbooks in mathematics. They too are frequently guilty of overemphasizing unimportant or obsolete material. It is also true that textbook writers are not always able to suggest newer and better things, and that textbooks are often made merely to fit the demands of certain state syllabuses which contain much obsolete and otherwise undesirable material. It is, of course, true that for many schools the textbook is and will continue to be the curriculum, but this does not obviate the necessity for pioneer work on the part of

progressive teachers and for freedom to supplement or rearrange the material in a book that is reactionary in treatment, is insufficient in its offering, or admits of improvement in sequence.

The makers of standardized tests in recent years have erred in including in their work certain exercises and problems that thoughtful teachers everywhere have no desire to see perpetuated in our schools. In fact many of these undesirable elements were obtained by the makers of tests from existing courses of study and from textbooks. Thus it is obvious that such tests cannot be used as the sole basis in the selection of desirable material.

We also know that it is not safe to try to determine what mathematics should be taught by merely counting the frequency with which certain mathematical terms are used in a few current editions of newspapers and magazines. Such a method is so unreliable that even those who have pretended to believe in it are abandoning it for more reliable criteria.

Finally, it is fair to say that we cannot satisfactorily formulate the course of study by going out in the world and asking individuals chosen at random what mathematics is useful to them. The fact is that not one of them ever knows just what use he has made of mathematics. Moreover, no one of them has probably given thought to the question of determining how he might have used mathematics profitably if he had only given the matter a little serious attention.

Any one of the above criteria may be of service in selecting the material for a course, but not one or even all will be sufficient for our purpose. If the objectives selected are to meet our modern needs, we must have at least one other criterion, — one that is at the same time the result of experience and of good judgment. This last criterion is the opinion of the most expert among the well-trained teachers of mathematics — those who are able not only to tell how they use the science but also to show how it may be used in the present and in the future for the betterment of mankind.

Method of Making the Selection. When people have to consult with experts, they usually get the opinion of the best they

can find. The material suggested in the list given later, in Chapter III, is the result of the combined judgment of many different teachers who not only know what is taught in the schools, but who are also the best qualified to speak with authority as to the needs of the new curriculum in mathematics in the training of the educated citizen. They are teachers of wide experience, themselves often trainers of teachers, and in several cases those to whom the latter have come for further preparation in their work. The importance of securing such a consensus of opinion cannot be overemphasized. To reach all such teachers is manifestly impossible; to reach carefully selected groups has been the purpose attained.

Importance of a Flexible Curriculum. It is needless to say that the material of the curriculum should not be allowed to crystallize too soon, if at all. It is much more important to keep it plastic and subject to the possibility of improvement. In other words, if properly conceived, curriculum construction is a continuous process. It is therefore important that we encourage careful discussion of all suggested plans for reorganization, that we weigh the opinions of the best writers, and that we avoid any spasmodic adoption of some suggested scheme as our final decision upon this important problem.

3. The Place of Method

Importance of Method. A Latin writer long ago remarked, "Poets are born, not made." In a way the same is true of teachers and, indeed, of those who succeed in any other walk of life. But however valuable inherent qualities of native ability may be, mere knowledge does not suffice; mere "knack" will not serve the purpose; the successful teacher must not merely be born, he must acquire knowledge of his subject and must profit by the counsel of experts in his chosen field. It is therefore important that we should consider with care the question of method.

Two Phases of the Subject. We should remember, however, that there are two sides to method, namely, methods of teaching

and methods of learning. We have done a great deal with the former in our normal schools and teachers' colleges, but it is only recently that we have begun to study seriously the psychology of learning. In later chapters we shall give more attention both to the methods of teaching and to the psychology of acquiring knowledge.

4. The Psychology of Learning

How the Pupil Learns. The practical problem in the psychology of learning is to find out how pupils learn most easily and economically, and we cannot expect to establish the most satisfactory curriculum unless we know the important results of this investigation. It is useless to say that we wish to teach certain subject matter when we know that this is so difficult that the pupils cannot comprehend it, for the time required to teach such topics is out of all proportion to their importance in the education of the ordinary citizen.

The Material for the Course. The material which will be suggested later is such as can be taught to children in the junior high school in the time ordinarily allotted to mathematics. It is material that has been tested in the classroom and is a necessary part of the equipment of every citizen. The teacher, however, should adjust it to the needs of his local group, keeping in mind the fact, as already stated, that the fundamental topics selected should meet with general approval throughout the country.

Proper Organization and Grade Placement. What has just been said implies that the material in the course of study should be effectively organized, arranged in convenient units, and properly placed with respect to the grades in which it is to be used. If a particular city or school undertakes to establish a curriculum which is not in harmony with any textbook that it can find, — usually a doubtful policy, — only those who are qualified by superior training, scholarship, and experience should be admitted to the committee in charge of the work. With all the variety of textbooks available, it is generally pos-

sible to find one that can be followed to better advantage than any local program that can be devised. With respect, however, to correlating the work to local needs and interests, the work of a curriculum committee is valuable.

5. A Testing Program

The Place of Tests. The last but by no means the least subject to be considered in determining the proper curriculum in mathematics is that of an adequate testing program. The purpose of such a program is to find by proper diagnostic tests whether the objectives previously stated are being fully realized. Without tests we cannot know whether the material is too difficult for the pupils. If the desired degree of mastery has not been secured, we wish to ascertain how good the results really are. The tests will reveal many things about the strengths and weaknesses of the pupils that cannot be as well discovered by any other means. In this way they become instruments for diagnosis and for the improvement of instruction.

Remedial Work. When such diagnostic tests are given and the results are poor, the teacher knows that his work was poor, that the pupils did not have the will to learn, or that the material was intrinsically too difficult. In any case, the teacher can use the results as a basis for such remedial teaching as he thinks necessary. If the pupils have not done their part, he is faced with a disciplinary problem; but if he decides that the material is too difficult, certain objectives may have to be modified.

Nature of Tests. No test can be condemned if it serves the purpose for which it is intended. We should be careful, however, to secure the best tests available and to use them to the best advantage. The greatest need in the matter of tests in the junior high school is for those which are diagnostic and which test what has been taught in the course. Such tests are also serviceable as teaching devices, supplementing the textbook, supplying material for drill upon the fundamental skills, and serving as measuring devices. A more complete discussion of the testing program will be given in Chapter XI.

QUESTIONS AND TOPICS FOR DISCUSSION

1. What is meant by saying that in curriculum-building "careful attention must be given to bridging the gaps between school and society, and between curriculum and child growth"?

2. In order to have an ideal discussion of any form of improvement in a curriculum in mathematics, what variety of interests and experiences should be brought together around the conference table?

3. Recent attempts to study curriculum revision have produced these results: (1) a historical review of curricula used in the past, (2) a description and evaluation of contemporary practices, and (3) a statement of foundation of principles for curriculum reconstruction. Discuss the importance and significance of each in the reconstruction or improvement of the mathematics curriculum.

4. What has the theory of "mental discipline" had to do with the curriculum in mathematics in the past? What is its present status?

5. What influence has been exerted on the curriculum in mathematics in the past century by extra-mural examination boards? What is their present influence?

6. Compare if available the syllabuses of the College Entrance Examination Board in algebra and plane geometry with the course of study used in these subjects where you last taught. If you have not taught these subjects, use some available course of study in each of these subjects.

7. The following five purposes or aims have been stated for the junior high school:

(1) To continue in so far as practicable and desirable, and in a gradually diminishing degree, a common integrating education.

(2) To ascertain and reasonably to satisfy the pupil's important immediate and assured future needs.

(3) To explore, by means of material in itself worth while, the interests, aptitudes, and capacities of pupils.

(4) To reveal to the pupils, by material otherwise justifiable, the possibilities in the major fields of learning.

(5) To start each pupil on the career which, as a result of the exploratory course, he, his parents, and the school are convinced is most likely to be of profit to him and to the State.

Show how any or all of the purposes above may be of use in determining a curriculum in mathematics for the junior high school.

8. Assuming that dynamic teachers are not always to be had, why should this necessitate a dynamic curriculum?

9. What is meant by saying that "no matter what the objectives, the textbook is the curriculum"? Does the truth of such a statement place any further responsibility upon textbook writers? If so, what is the nature of this responsibility?

10. What kinds of scientific studies might be made that would be helpful in determining a better curriculum in mathematics for the junior high school?

11. Discuss the statement frequently made that "junior-high-school textbooks in mathematics are for the most part a handing down of the conventional high-school material."

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CHAPTER II

HOW TO BEGIN THE COURSE IN MATHEMATICS

1. Various Plans

The Main Purpose in Arithmetic. It should never be forgotten, and it will be repeatedly stated in this work, that the main purpose in teaching arithmetic is to train children to compute accurately and with reasonable speed in those operations which they will probably use in later life.

This statement may seem to be self-evident and therefore unimportant, but it is neither. A large part of our inherited arithmetic is little concerned with computation, and in Grades VI–VIII many teachers pay little attention to the operations that the children will need in their daily lives. It is frequently the case in courses of study, both state and city, that they require the teaching of operations that are so rarely used as to be of little value to the great majority of people; and they often tend to perpetuate obsolete processes, obsolete business methods, and obsolete terms, rules, and definitions that have no place in the modern school. What is even more serious in its disastrous consequences in our schools is that many of the tests, which should tend to raise the standard of the work, often lower it through their use of obsolete and useless material, thus blocking the very progress which it is their duty to encourage.

What the Pupil Knows. At the beginning of the seventh grade pupils often do not have a mastery of the fundamental skills of the previous grades. In fact, only a few of the objectives are realized to as high a degree of mastery as 90 or 95 per cent. Nearly all pupils will know that there are 12 inches in 1 foot, but very few will know how to divide 31.549 by 3.8, an operation of relatively little importance to most people but one that is, nevertheless, taught in most schools if not in all.

Even in the case of material of undoubted value certain studies have shown that the schools are securing less than a fifty-per-cent mastery of many of the items. Indeed, as to many of the fundamentals in which such poor results are being secured it seems evident that the little which pupils assimilate is not due so much to the influence of the teacher as to that of the home or environment. Simple games, for example, have probably given to the pupils a very appreciable part of their knowledge of elementary number relations. Indeed, the degree of mastery of certain things in arithmetic is so low that one would expect as good results from pupils of fair or superior ability who have not studied the particular topic at all. In such cases the children would obtain their knowledge from such contacts with the arithmetic of daily purchases and the like as would come within their understanding.

What Causes Such Poor Results. Such a situation as that described above is due in part to the attempt to teach too many things, but it is due also to other factors. One of these is the lack of a clear understanding on the part of both teachers and pupils as to what objectives are fundamental and need to be attained. Another factor is that there are few teachers who are careful to make certain that the proper degree of mastery is attained. This can be done only by giving well-planned diagnostic tests.

As evidence of the inefficiency resulting in large part from wasted energy, in a group of over three thousand pupils in Grade VII it was recently found that only fifty-eight per cent knew the number of feet in a mile. The knowledge of this relationship is not, to be sure, of supreme importance; but it is taught in every school in the country and has some value as a piece of general information, and so it might have been thought that many more than half the pupils would have retained it. It seems evident, therefore, that the returns upon matters which are considered important enough to teach are far from satisfactory under present conditions.

The difficulty of expressing a definite opinion with respect to the degree of mastery to be expected lies in the fact that we have

no consensus of opinion as to the types of mastery to be attained in the various arithmetic objectives which we pretend to attain in our schools. This does not mean, however, that we should not attempt to decide upon what mastery we should expect.

What Such Results Mean. The fact that we do not obtain the degree of mastery that we might naturally anticipate does not mean that we need to confine the work in the mathematics of the junior high school to arithmetic alone. It is further evident that it is not at all safe to discontinue entirely the study of arithmetic at the close of Grade VI. Certain of the fundamental skills and valuable pieces of information must be incorporated in the later work if we are to retain them until the time when, on leaving school, the pupil will have need for them in practical life.

Complete mastery or a satisfactory degree of mastery will be more nearly realized if the teachers in the elementary school will center their attention on fewer and more fundamental processes. Even if complete mastery were realized before the seventh grade, however, we should still need to give more training and experience in certain of the applications of arithmetic which the citizen should know, if we wish him to continue to make intelligent use of his knowledge.

In any event, it is not at all necessary to give eight years to the study of arithmetic alone. For this and for other reasons, it is as important to give as careful attention to the curriculum in mathematics in the seventh and eighth grades in places in which no junior high schools are organized as in those in which the organization follows the modern plan. In other words, the course herein set forth should be generally taught even where Grades VII and VIII are not part of a junior high school.

Arithmetic is finished at the end of the sixth grade in practically all countries except the United States, in so far as the fundamental operations or essential parts of computation are concerned. There are, of course, very many practical applications which are too hard for pupils to understand in the sixth grade. We shall need some of them in the junior high school, but they will not include, as was once the case, either cube root or thirty cases of percentage.

Advantages of Teaching Arithmetic First. There are good reasons why the work of Grade VII should begin with an extension of the arithmetic previously studied. In the first place the pupils are more or less familiar with the fundamental arithmetic skills and their applications to situations in daily life. Secondly, they do not look with doubt upon a subject with which they are already familiar, and it has always been found advantageous to base any new work in a subject upon the pupil's previous experience.

What to Teach First in Arithmetic. We have considerable evidence to support the belief that it is not a good plan to begin the work in arithmetic in the seventh grade with any very extended formal reviews of the fundamental skills and processes that have been developed in the first six grades. Long reviews are almost sure to be wearisome to the pupils. They think they know their arithmetic, and even though this is not true it is better to develop their skills by some more appealing method than that of merely repeating at great length what they have already had. With a few brief reviews of matters of computation there should be given certain important applications of arithmetic which involve the use of such skills as the teacher thinks necessary; and if the pupils find out that they do not know the arithmetic, they then have a definite motive for review. For example, a pupil may think that he knows his arithmetic until he comes to certain problems in the study of the geometry of size. He may then discover a need for carefully reviewing certain fundamental numerical operations and skills.

The reader will find one method of beginning the work of the seventh grade in arithmetic by studying the model lesson outlined on pages 320–324.

Disadvantages of Teaching Arithmetic First. There are, however, certain disadvantages in such a plan. The pupils are often tired of arithmetic, especially if they have had too much mechanical drill; they welcome a change just as adults do when they seek variety in their work. Moreover, if the pupils have not obtained good results in their arithmetic work, they feel that they should like to try something new.

On this account teachers often find it preferable to start with a new topic like intuitive geometry, making the review of the arithmetic a part of this work. When the pupil thus comes to feel the need for it he will be more ready to resume his study of arithmetic and will see that it is a very essential part of his new equipment. In general, with any usable textbook it is possible to rearrange the sequence to provide for beginning in such a way as to meet the special needs of any particular school.

Intuitive Geometry may come First. In harmony with the purposes stated in the preceding chapter it is desirable for the pupil to have some experience with types of material which he is likely to need if he leaves school at the end of the seventh grade. It is now generally recognized that everyone should have some knowledge of the essential facts of intuitive geometry. This is the part of geometry in which the pupil looks at a figure and says that certain things are true because he cannot conceive of their being otherwise. The argument for having this subject precede the study of arithmetic in the seventh grade is, as already stated, that the pupil likes variety, and the very nature of intuitive geometry makes it a welcome change. Moreover, as experience with the children in such work shows, the subject is well within the grasp of seventh-year pupils. In some circumstances the subject is quite as much a part of the pupil's daily needs as arithmetic, especially, for example, in shop work.

The First Lesson in Intuitive Geometry. Wherever intuitive geometry is taken up in the seventh grade, the first lesson should be one that will at once arouse the interests of the pupils. If logical order were followed, one might begin with the geometry of position. It is likely, however, that most teachers will prefer to begin with the geometry of form. Even if this topic is selected, there are several ways in which to begin. We may begin the work, as many teachers do, with a discussion of primitive methods of shelter, or we may start with a simplified discussion of other spaces than ours, starting with pointland and running through the extensions of lineland, flatland, and so on.

To make the matter more definite the outline of one of the various methods is given on pages 308–314. This lesson has

proved to be very interesting and enjoyable to pupils, and it provides a basis for a discussion of the geometry of form.

Disadvantages of Beginning with Intuitive Geometry. The objection most frequently raised by teachers to the plan of beginning with intuitive geometry is that if such a plan is followed, the pupils will not wish to return to the study of arithmetic. The answer to this contention is that the reaction of the pupils to this plan obviously depends upon the way the return to arithmetic is motivated. If the shift is abrupt, formal, and uninteresting, the result is obvious. If, however, the teacher is careful not to return to arithmetic until he has shown the pupils that they have great need for a better knowledge of certain numerical work, then the result is sure to be different.

Algebra may come First. Because algebra is generalized arithmetic, some teachers have considered it good policy to introduce algebra at the beginning of the seventh grade, but this has not met with general approval. While the study of the subject is important in the education of the average citizen, it is more difficult than the arithmetic and intuitive geometry which are commonly given to pupils in this grade. If any algebra at all is offered in this school year, it should consist of a brief introduction to the use of the formula, the first four simple types of equations illustrated on pages 347–348, and the graph. This much of algebra could then be brought in naturally under that part of the treatment of the geometry of size in which rules for measurement are given. The argument for such work in the seventh year is that if algebra is spread through Grades VII, VIII, and IX, as in the case of most European schools, the final effect on the pupil will be better than if the subject is confined to a single year.

The First Lesson in Algebra. Inasmuch as the main use of algebra centers around the formula it would seem advisable to begin the course with a study of this phase of the subject. The reader will find on page 302 an outline of a first lesson on this topic — one which has actually been used in a seventh-grade class, although it would come in the ninth grade as the work is arranged in many schools.

2. Recommendations

How to Begin the Course. The question as to the precedence of topics in the work of the seventh grade can probably best be left to the judgment of the individual teacher, who can reach a decision based upon the peculiar needs of the local group of pupils. If the previous work in arithmetic is excellent or if the pupils need a change, it may be advisable to start with intuitive geometry. Some care must be taken to have the general practice of the country uniform with respect to the year's offering as a whole on account of the mobile character of the school population, but this need not affect the sequence within the grade. Finally, this question of sequence is of much less importance than that of making the work interesting, inspiring, and genuinely useful, no matter what order is followed.

Other Schemes. Certain teachers feel that arithmetic should be placed last in the junior high school because it is so hard. This argument has little weight, however, because arithmetic, like any other subject, is difficult or not according as we make it so. Moreover, it is not practicable to postpone it to Grade IX for the reason that a large percentage of those who enter Grade VII drop out at the end of the year, or in the year following, and thus would miss that part of mathematics which they need the most.

Other teachers assert that arithmetic should be scattered all through the course. But this is unsatisfactory unless the correlation is very close, which is usually not the case. There is always a certain amount of mixing of topics that is legitimate; for example, arithmetic mixes well with a certain part of intuitive geometry and with a relatively small part of algebra. But the part of arithmetic that correlates well with these subjects is not the arithmetic of life, such as the arithmetic of thrift, and so it is quite evident that we shall obtain the best results if we teach arithmetic largely by itself, and then have the work in measuring merge naturally into intuitive geometry, finally combining both of these with the early use of the formula.

Grade Placement. While no single order of topics or particular grade placement of materials is essential, there should come to be a fairly uniform practice in the placing of the larger units within the various years. The following outline suggests what has proved in actual practice to be a successful method.

Grade VII	Grade VIII	Grade IX
Arithmetic	Algebra	Algebra (beyond mere utility)
Intuitive geometry	Business arithmetic	Numerical trigonometry
Algebra	Review	Demonstrative geometry (a brief introduction)

Results of Such a Scheme. No matter how the material is arranged, if we follow this general scheme, we may expect three important outcomes. First, the pupil will have the best possible preparation, within the time allowed, to meet the mathematical needs that confront him in practical life. Secondly, he will be preparing himself in the best possible way to continue his work in mathematics and science if he proceeds further in school. In the third place, the pupil will have the door of mathematics opened, so to speak; he will have a chance, by exploring its various branches, to find out what the science means; and he will thus be enabled, with the help of his parents and teacher, to determine whether he should continue studying mathematics or should devote his time to other branches.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Discuss the advantages and disadvantages of beginning the course in mathematics in the junior high school with arithmetic; with intuitive geometry; with algebra.

2. Why should the mathematics of the seventh, eighth, and ninth grades be generally the same in all types of school, whether organized on the junior-high-school plan or not?

3. What are the differences, if any, among the degrees of mastery which we should expect to attain in the various objectives in teaching arithmetic? in teaching intuitive geometry? in teaching algebra?

4. Are the schools attempting to attain greater mastery of certain topics than it is reasonable to expect? Give a few reasons to support your answer.

5. Discuss the possible advantages in beginning the study of intuitive geometry with the geometry of form; with the geometry of size; with the geometry of position.

6. State two or more desirable ways with which to begin the course in arithmetic in the seventh grade. Show why you prefer one of these above the others.

7. State a reasonable argument for giving in Grade VII a brief introduction to the study of algebra, say, simple formulas, equations of a simple type, statistical graphs, and graphs of a few simple mathematical laws; this being followed by a half year of algebra in the eighth grade, and the rest of elementary algebra in the ninth grade.

8. State any particular advantage that you see in the fact that the National Committee on Mathematical Requirements suggests (pp. 29–30) five different plans for the course in junior-high-school mathematics.

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CHAPTER III

OBJECTIVES TO BE ATTAINED[1]

1. THE PURPOSE OF THIS STUDY

General Purpose. No work is ever completely efficient unless the object in view is clearly understood. The great object in studying mathematics in the junior high school is to give to pupils some idea of the general nature and uses of business arithmetic, intuitive geometry, practical algebra, and the simplest part of trigonometry, together with a knowledge of the meaning of a demonstration. All this requires on the part of the teacher the recognition of subsidiary objects, which are generally classified as objectives. It is the purpose of this section to consider from the mathematical standpoint certain important and typical objectives.

Classification. Five ways of listing objectives were given in Chapter I. It was also stated that the order here adopted would be the one which experience has shown to be the most helpful to teachers, even at the expense of some duplication of statements and of some sacrifice of logical sequence.

It should, however, be repeated that the list is purposely incomplete. It would be a very simple matter to give a thousand or more objectives, but such a list would be lacking in emphasis and would be so wearisome in its details as to repel instead of assisting the teacher who is seeking to obtain a clearer view of the purposes in teaching mathematics in the junior high school. Every thoughtful teacher will mentally add to the list, and some teachers will, and should for their purposes, eliminate such items as fail to commend themselves to their best judgment.

[1] Portions of this chapter appeared in the Second Yearbook of the National Council of Teachers of Mathematics, New York, 1927, and are published in the present work with the permission of the editorial board.

2. Great Central Mathematical Objectives

Mathematical Objectives. Certain objectives are strictly mathematical, while others are of a more general nature. The former may be classified as follows:

A. An Introduction to the General Range of Elementary Mathematics

1. *The application of arithmetic to business problems.* This means that the pupils should be shown the uses of arithmetic that any well-educated citizen is expected to understand, but not the technicalities of special branches like banking, bookkeeping, or machine-shop practice.

2. *Intuitive geometry.* This means such a knowledge of shape, size, and position as people need to have for purposes of general information.

3. *The algebra of the formula, graph, directed number, and equation.* This means that these four concepts are so important in elementary science, in simple mensuration, and in ordinary business that everyone should know something about them.

4. *The general nature of trigonometry.* This means that all pupils should have some idea of how distances and heights are measured by this simple device. It does not mean that any of the difficult parts of trigonometry are to be studied in the junior high school.

5. *The significance of a demonstration.* This means that every pupil should have the privilege of seeing the nature and of understanding the significance of a mathematical proof. This is best done by a few theorems in geometry. It does not mean that the pupil is to take a difficult course in demonstration, but that he should know the force of the word "demonstrate."

B. Some Appreciation of the Power of Mathematics

1. *In ordinary life.* This means that the pupil should appreciate something of the power of computation, of its application to common measurements, of the power of the formula to "do things," and of the value of the graph in everyday business.

2. *In related fields of knowledge.* This means that the pupil should see that without mathematics there could be no modern business, no engineering, no machinery beyond the simple lever and the wheel, no currency, no insurance, no travel in the modern sense of the word, no great buildings, and no sciences. He cannot be brought to see all the applications of mathematics in these fields, but he can be led to see enough to show him the enormous power of the subject.

3. *In further mathematical work.* This means that we cannot progress in mathematics without the elements upon which we depend. Men do not become great merchants, bankers, or scientists without a knowledge of the mathematics of the junior high school, acquired there or in other ways. Great tunnels are planned, great bridges are built, and great ships sail the seas, only as the result of the knowledge of higher mathematics, and this knowledge is possible only after the earlier mathematics has been mastered.

C. The Increase of Certain Powers

1. *Using symbols.* This means that, although the pupil has long been using number symbols (1, 2, 3, $\cdots$; I, II, III, $\cdots$) and such signs as $+$ and $-$, the power to use mathematical symbols must be extended to include other forms (a, $\sqrt{b}$, $-n$, x^4, etc.) if any further progress is to be expected either in pure or in applied mathematics.

2. *Analyzing relations.* This means that the pupil should be trained to select the essential facts from among those which are not essential to the solution of a problem; that he should connect these in a logical manner; and that he should so use them as to attain a result that he can depend upon as accurate. This power is fundamental in all scientific and economic fields.

3. *Constructing graphs.* This means that the use of graphs has now become so common that everyone should have some idea of how to represent simple statistics by a bar graph or a curve-line graph, some of which work may have been done in the earlier grades. It does not mean that everyone should know how to construct some of the more complicated types.

4. *Interpreting graphs.* This means that people generally need to know how to interpret or to find the meaning of such graphs as are ordinarily seen in newspapers, magazines, government reports, and popular scientific journals and books. These include not only bar and curve-line graphs but circular graphs and other types that are too difficult for the pupils to construct.

D. Fostering the Study of Mathematics

1. *For the improvement of more advanced mathematics.* This means that such pupils as show a taste for the subject should be discovered and then be encouraged to pursue their studies farther. Most parts of advanced mathematics have practical applications, and all are concerned with the discovery of truth. The possible use of all this work cannot be foretold. It does not mean, however, that pupils should be forced to study any of the more advanced parts of mathematics for which they have manifested no taste.

2. *For mental pleasure.* This means that mathematics offers mental pleasure to a great many pupils. There is the same reason for gratifying this taste as for gratifying a taste for literature, music, or the fine arts. It does not mean that any prolonged attempt should be made to compel a pupil to like what is repugnant to him.

General Objectives. As already stated, certain objectives are of a general nature instead of being strictly mathematical. The attainment of these objectives, however, is facilitated by the study of mathematics. The objectives themselves may be classified as follows:

A. Establishing Certain Habits

1. *Neatness and method.* This means that the pupil's written work should be neat, uniform in method of presentation, and clean in general appearance. It should suggest to him the desirability of neatness and cleanliness, and of the methodical arrangement of all his work. The very nature of mathematics lends itself to this type of habit formation, this being the one science which demands with great emphasis these several features.

2. *Thinking.* This means that mathematics should afford and does afford an unusual opportunity for concentration and for the play of constructive imagination. This is considered further in a later chapter.

3. *Moral conduct.* This means that mathematics affords a constant opportunity of displaying honesty to oneself. The pupil should constantly ask, "Is this unquestionably correct? Have I checked this operation? Am I honest to myself in being positive of this result before I proceed farther and cause myself trouble through my error at this point? Can I rely upon myself and then lead others to rely upon me?"

4. *Character.* This is an outgrowth of the preceding paragraph. It means that mathematics constantly affords opportunities to encourage a reverence for truth, for absolute accuracy of statement, for beauty of form, and for the recognition of the unity of "the good, the true, and the beautiful."

B. Exercise in Fundamental Modes of Thought

1. *Simplicity of language.* No science lends itself to the cultivation of simplicity of language to the extent found in the case of mathematics. The whole range of this branch of knowledge is characterized by succinctness of expression and the elimination of complex phraseology. In elementary mathematics there has been a special effort made in recent years to avoid the use of unnecessary technical terms and of pedantic expressions of all kinds. The effect of all this upon the pupil's style of speaking and of writing cannot fail to be salutary.

2. *Accuracy in reasoning.* In the junior high school the pupil becomes more conscious of the fact that one of the essential features of mathematics is the accuracy of its processes and of the reasoning employed in its solutions. Nowhere in the entire curriculum is such accuracy essential, and nowhere else does he receive such training in this phase of mental activity.

3. *Originality in thought.* This is found in connection with all branches of knowledge. A pupil may show originality in literature, music, historical interpretations, and science just as well as in mathematics. It forms a definite objective in every line

of thought. It is therefore of necessity a definite and important feature in the junior-high-school work in mathematics. The pupil should be encouraged to use all the originality he can in attacking solutions of whatever problems are presented. To compel a pupil to solve a problem as the teacher directs or as the textbook solves it is to stifle a pupil's originality. If his plan is not as economical as some other, the better one may be shown to him afterwards.

3. Psychological Sequence of Objectives

General Purpose. Having considered the great central objectives in the teaching of junior-high-school mathematics, we shall now suggest a list of certain important detailed objectives, arranging them with some attention to their psychological sequence.

A. Appreciation of Mathematics as a Useful Art

1. *Power of expressing data systematically.* This means that from certain facts or data ("things given") that are available for the solution of a problem, we shall be able to select the relevant data, tabulating them in the most systematic way by which to express the facts of the case and to make the interpretation simple.

2. *Scrutinizing these data in solving problems.* This means that we should be careful to find whether the one who gathered the facts was reliable; whether he has used the best methods (graphical or otherwise) for arranging them; and whether by further examination we cannot make some improvement upon his work.

3. *Organizing these data as an aid to memory.* This means that, by means of graphs or otherwise, we should learn how to develop methods for making certain data stand out prominently in our minds whenever it is important that such data be immediately recalled. For example, it may be necessary to remember certain relative values which, when represented by curvilinear graphs, will be more easily remembered than the numbers upon which they are based.

4. *Succinctness of mathematical statements of laws as formulas.* This means that the pupil should appreciate the great value of the formula as a practical substitute for the long rule that was formerly used. For example, to state that $A = \frac{1}{2}\,bh$ is to use an expression that is far more succinct and useful than the old rule for finding the area of a triangle.

5. *The equation as an aid in using formulas.* This refers to the fact that from the interest formula $i = prt$ we can readily obtain a formula for p, r, or t by the ordinary methods of very simple equations. Therefore, from this single formula we can easily find three others, this being much better than the old plan of learning four long rules. This is the chief value of an equation, — not the solving of a long list of number puzzles.

B. Appreciation of Mathematics as a Science

1. *Significance of symbolism.* This means that the pupils should see what a powerful tool we have in the symbols of arithmetic and algebra, as in the significance of the expressions $a^2 + b^2 = c^2$ and $a^2 = c^2 - b^2 = (c + b)(c - b)$.

2. *Interlacing of branches.* This means that there is no part of mathematics that is not related to all the other parts. For example, the equation $a^2 + b^2 = c^2$ is a statement that is related to arithmetic, algebra, geometry, and trigonometry.

3. *The relation of mathematics to allied subjects.* This means, for example, that the subject of latitude and longitude in geography is closely related to intuitive geometry, arithmetic, trigonometry, and analytic geometry; that a subject like physics could hardly exist without arithmetic, algebra, geometry, and trigonometry, and that it also makes extensive use of higher mathematics; and that all the other sciences are constantly indebted to mathematics of various kinds.

4. *The relation of formulas to general truths.* This means that many laws which are not directly concerned with algebra, for example, are now expressed in algebraic language. For example, in any crystal the number of faces plus the number of vertices is equal to the number of edges increased by 2; that is, as a formula, we have $F + V = E + 2$.

5. *The eternal verities of mathematics.* This means that a correct mathematical statement is true "yesterday, today, and forever." A human law may change, every building must some time decay, all life is continually changing, but $(a+b)^2$ is always equal to $a^2+2\,ab+b^2$.

6. *The universality of functional relationships.* This is a common but unnecessarily difficult way of saying that everything in this world depends upon something else, just as the area of a circle depends upon the length of the radius, the length of the diagonal of a square depends upon the length of a side, and the distance traveled is a function of the rate and the time. For example, the height of a tree depends upon the kind of tree it is, and also, for a certain period, upon its age.

7. *The value of mathematics for its own sake.* The French have an expression *l'art pour l'art* ("art for art's sake"). This means that while painting may be useful on a barn, a painting of a landscape may be beautiful and may be admired for its artistic qualities. So geometry has various uses, but we may like it even more for its beauties, its standards of truth, and its succinctness of statement.

8. *Relation to the Infinite.* This means that as the pupil progresses he will see through mathematics an approach to some conception of the Infinite that he can never see in any other field of study.

9. *Recreational side of mathematics.* This means that mathematics has its recreational side if only we will seek it out. It is no mere chance that a "magic square" like this one exists. In this case each row, each column, and each diagonal has 15 for its sum. This magic square is probably the oldest record that we have of mathematics in Eastern Asia. All mathematics is a kind of game to those who know how to play it.

8	1	6
3	5	7
4	9	2

10. *Connection with art.* This means that mathematics enters into all great architecture; into a large part of decorative art, as in symmetry, spiral forms, ellipses, and regular polygons; into music; and even into painting, as in the study of perspective.

11. *Rhythm of mathematics.* This means that through all mathematics there runs a rhythm. A child likes to say "*5*, 10, *15*, 20, *25*, 30," or "*2*, 4, *6*, 8, *10*, 12," and the pupil in algebra takes pleasure, if encouraged to do so, in the expansion of $a + b$ to the various simpler powers.

12. *Relation to nature.* This means that, if the idea is suggested to him, the pupil will tend to take an interest in seeing how constantly mathematics enters into natural forms. The cross section of a banana, an apple, or the seed of a rose will serve to initiate him into a search for such relations.

C. Appreciation of the Historic Growth of Mathematics

1. *Mathematics a moving stream.* The history of mathematics as a separate study is not desirable in the junior high school, but the teacher will add greatly to the interest in the subject by referring to the fact that mathematics started as a tiny brook in the remote mountains of the Past and that it has increased in power as it has watered the lands all through the centuries.

2. *Significance of our numerals.* All pupils should see the power of our numerals to express large numbers readily; they should know that these numerals are only about a thousand years old in Europe, and that they probably came from India by way of Arabia; and they should also know that there have been many other systems, out of which we still keep only the Roman.

3. *Growth of the fraction.* This means that pupils should know that for some centuries our "common fraction" was, as its name says, common all through Europe and America, but that today the decimal is the more often used, as in fractions of a dollar. They should know that we now use only halves, thirds, fourths, and eighths in most of our everyday business. The decimal fraction, for practical use, is only about a hundred fifty years old, although it was known considerably earlier.

4. *Displacement of compound numbers.* It is a valuable thing for the teacher and an interesting one for the pupil to know that until very recently such compound numbers as 10 mi. 32 rd. 4 yd. 2 ft. 8 in. were taught and used. About all

that we have left in common use today is found in cases like 2 ft. 8 in., 1 lb. 4 oz., and 2 hr. 45 min. The compound number has been almost entirely displaced by the common and decimal fraction.

5. *Merging of decimals and per cents.* This means that there is no numerical difference between 6% and 0.06, and that nowadays per cents are treated as a part of decimals, or immediately in connection with them, by translating the first form into the more convenient 0.06.

6. *Systems of measure.* This means that the pupils should see that the world discards continually materials that have outgrown their usefulness. A hundred years ago people used candles, they traveled long distances on horseback, they measured land by roods, they measured cloth by ells, and they used the "long hundredweight." We now use fewer measures, and a large part of the highly civilized world uses the metric system. For general information we need to know about the meter (radio wavelengths), the kilowatt (electric meters), the gram and the cubic centimeter (medicine), the kilometer (international sports), and, in general, about the metric system in comparison with the older and more difficult one which we generally use.

7. *Change from rule to formula.* This means that we should all see the value of the improvement made in the last few years in changing the emphasis from the long and often difficult rule to the brief and simple formula.

8. *Development of symbols.* This means that we could have no formulas without symbols, and that our ordinary symbols of algebra were invented to meet world needs about three centuries ago, thus revolutionizing mathematics and making possible such work as that of the junior high school.

9. *Growth of applied mathematics.* This means that the world has always been applying its mathematics, and that these applications have grown with world needs. No one thought, when trigonometry began to be well known in the schools about three centuries ago, that it would be used today to find the size of our universe or that it would be made so simple that we could teach it in Grade VIII or Grade IX.

10. *Great names and great periods.* This means that just as we know who William the Conqueror was, so we should know that Pythagoras was also a great conqueror, but in another field, and that his story is quite as interesting as that of the Norman who subdued England; that it is a good thing to know about Washington as the father of his country, and also about Euclid, who was the father of elementary geometry; and that the Elizabethan period means much to the world, but that the great period in which geometry developed on Greek territory means much more in the history of civilization.

D. Attitudes of Mind to be Developed

1. *Responsibility for accuracy.* It is not merely a habit of mechanical accuracy that we develop but also a feeling of personal responsibility for accurate reasoning and accurate results.

2. *Satisfaction with thorough work and precision of statement.* It is sometimes asserted that our great industries tend to make men but little more than parts of a machine and that the pride of achievement is lost. This is not entirely true; indeed the amount of truth in the statement is probably less than we think; but at any rate it is not true in the intellectual life of the school. There is such a thing as just pride and honest satisfaction with a good piece of work, and nowhere can this be better fostered than in the study of mathematics.

3. *Common-sense estimates of results.* This means that a pupil should early cultivate an attitude of mind that leads him to make a sensible guess as to a result before he begins his solution of a problem in arithmetic. There is no better rough check upon the accuracy of his work.

4. *Dissatisfaction with vague results.* For example, when called upon to interpret the meaning of a graph, a pupil should feel dissatisfied if he has not something reasonable and positive to offer; and when he solves a problem, he should not be satisfied with a result that is not clearly expressed.

5. *Recognition of irrelevant data.* It is one of the valuable features of geometry, even of the intuitive kind, that a considerable number of things are known about a figure, but that out of these

we need to use only a few to prove what we wish or to compute a required length or area. The attitude of mind that leads us directly to the rejection of unnecessary facts is a valuable asset to anyone.

6. *Discrimination between the true and the false.* This means that in mathematics we constantly meet with the question "Is this true or is it not?" It is a healthy attitude of mind to develop, — that of weighing the two sides of every question of this kind.

7. *Desire to analyze a complex situation into its components.* This means that we should cultivate the habit of thinking of the details that make up the mass. For example, a graph may rise violently between two points; in interpreting the graph we need to state the reasons for this rise, and a large number of possibilities suggest themselves. It is a good attitude of mind that leads us immediately to analyze this mass of possible causes into its component elements and be able to select the most powerful ones.

8. *Self-reliance in attacking a problem.* There is a great difference between self-conceit and self-reliance. Mathematics should cultivate that attitude of mind that leads a pupil to say, "I not only *can* do this thing, but I *will* do it."

9. *Desire to search out the truth.* An old philosopher once said, "No pleasure is comparable to the standing upon the vantage ground of truth." It is a great asset to feel this and to have an attitude of mind favorable to the search for those things which are true. Mathematics is one of the foremost sciences in fostering an attitude of mind that always searches out the truth, — not merely the probable, but the actual.

10. *Constant seeking for applications of mathematics in daily life.* Such an attitude of mind will go far to awaken and to maintain an interest in all lines of mathematics that the pupil studies. It will be a constant source of surprise and interest to find out how many such applications we can see if we but search for them.

11. *Interest in developing skill in mathematics.* This means that in teaching we should endeavor to develop a spirit similar to that which leads a boy to take pride in his skill in tennis, a

girl in basket ball, and an adult in golf. We can hardly hope to do this to the same degree as in the case of ordinary outdoor sports, but we can approach the same objective.

12. *Desire to generalize.* This means that we should seek to establish that attitude of mind which leads to the discovery of general laws governing a range of special cases. This is illustrated in the case of the sum of the angles of a triangle. What is the sum in the case of a four-sided figure? one of five sides? one of six sides? — and so on. What is the general law? Does this hold in the case of a two-sided polygon? There are few things in geometry that give greater pleasure than the joy of generalizing.

E. Ideals to be Cultivated

1. *Devotion to truth.* This means that mathematics is concerned not only with proving that a statement is true but with discovering truths for ourselves. In the preceding paragraph a question was raised about a two-sided polygon, and the pupil who discovers that it really exists, that the sum of its interior angles is $0 + 0$, and that it obeys the laws of other polygons has made a step toward devoting himself to the search for truth.

2. *Originality in action.* This means that one ideal which every pupil should seek to cultivate is that he may depend only upon himself. If this is done early in mathematics, a pupil will soon come to feel an independence that will lead him to pay little attention to book proofs and to depend more and more upon his own originality.

3. *Neatness in solutions.* This means that good taste should be an ideal in our written work as it should be in our dress and in the furnishing of our rooms. People of good sense and refined taste do not care for a vulgar display of wealth in such matters, but they appreciate neatness. In our work in mathematics slovenliness in writing, in computing, and in the drawing of figures usually means slovenliness in thinking.

4. *An appreciation of our relation to the universe.* This means that it is chiefly through mathematics that we attain some grasp of this relationship. It is mathematics that tells us much of what we know of the infinitesimal as it is typified in the electrons in an

atom, and it is mathematics that reveals to us the grandeur of the infinite as typified in the space about us. Fortunate the class that has a teacher filled with such ideas and ideals and who is able to pass these on to the pupils without wasting time or making drudgery out of a great epic.

5. *Respect for one another as small portions of the infinite.* This means that we all have a feeling of infinity and of ourselves as part of some stupendous whole, and that mathematics cultivates this tendency. Such ideals make for the brotherhood of man, and pupils are apt to appreciate this if the fact is not made so dull as to become mere drudgery.

6. *Regard for the beautiful.* This means that, for example, the ideal of the beautiful exists in intuitive geometry as really as it does in nature or in the fine arts. It is a dull class that cannot be inspired to admire the beautiful in this subject, if only the teacher also admires it but does not talk too much about it.

7. *Loyalty to the family, the community, and the State.* This means that thrift is not the acquiring of wealth for ourselves alone but that it is inspired by a feeling of loyalty to the family, a feeling that old age must not make itself a burden; that insurance is not a burden but that it is an evidence of loyalty to the community, each contributing to another's loss; and that taxes are simply an evidence of loyally repaying the state for protecting us and our property, for educating us, for looking out for public health, and for providing good roads and pure water.

8. *Respect for a good reputation.* This means that in the study of banking and commerce, of investments, and of whatever has to do with care and honesty there should stand before the pupils the ideal of the good citizen, the man with a reputation that makes him trusted by his fellow men.

4. Relation to the Seven Cardinal Objectives

Seven Cardinal Objectives. It has been said that there are seven cardinal objectives in education: (1) health, (2) command of fundamental processes, (3) worthy home membership, (4) vocation, (5) civic education, (6) worthy use of leisure,

and (7) ethical character. These do not include the religious objective, which, so far as the public schools are concerned, is generally left to the church and the home. With each of these objectives mathematics has a certain degree of relationship, — in some cases close, in others not so pronounced. It would be possible to claim even closer relationships with some of these objectives that are set forth below, but only those bonds will be mentioned to which most people would agree.

Relation to Health. This is seen in the following topics:

1. Graphs of temperature, pulse, and respiration, and their interpretation.

2. Graphs in reporting the progress of wounds, and their interpretation.

3. Mathematics in the training of nurses.

4. Determination of the proper seating-capacity of schoolrooms and of the cubic space to be allowed per person.

5. Graphs of heart action, used by heart specialists.

6. Mathematics of optometry, — the measurements needed in correcting the vision.

7. Saving of time and energy through calculating-machines and tables.

8. Graphs of pupils' ages and weights, and the study of the records from time to time to determine physical condition.

9. Measurements of muscles, of hearing, and of strength.

10. Measurements of food values by calories.

11. Nature of disease as generated by the rapid multiplication of germs in geometric series until, after "exposure," the number of germs becomes so great that the patient "comes down" with the ailment.

Relation to the Command of Fundamental Processes. Taking fundamental processes to mean such attainments as reading, writing, speaking, computing, reasoning, and other like activities, mathematics is related through the following objectives:

1. To read and write numbers and to grasp their meaning in various situations; for example, in geographical statistics, business statistics, accounts, budgets, wage scales, and in a multitude of other cases in daily life.

2. To compute accurately and with reasonable speed. Without this ability on the part of people generally, civilization would relapse into barbarism. This involves the following items:

a. Drill for mastery on the fundamental operations.

b. Oral and mental work where the answer is estimated in advance and the results are approximated.

c. Desirable short cuts and rapid calculations.

d. Use of well-constructed practice exercises.

e. Checking of results.

3. To reason accurately, as in geometry and in problem solving, and to see that this same type of mental activity is transferable to life situations.

4. The ability to select the essential facts in a given situation and to arrange them logically, as is done in a problem in arithmetic or in a proposition in geometry. The question of the transfer of this power to life situations depends upon the teacher; if the transfer is sought it is easily secured.

5. Precision of statement, — an essential feature of mathematics. Like logical arrangement, it is readily transferred to literary productions if the teachers encourage pupils in the attempt.

6. The habit of concentration upon a problem.

7. The arrangement of an argument, like a proof in geometry, so that each statement follows logically from those which precede. Many successful speakers arrange their notes like a geometric demonstration.

Relation to Worthy Home Membership. The question that arises in this connection relates to those attainments which make the individual a worthy member of a home, and to the connection between these and his mathematical attainments. The relationship is seen in such objectives as the following:

1. A knowledge of bills, receipts, discounts, rent, wages, household expenses, house plans, as applicable to personal, family, and business accounts; to personal and family budgets; to rent, insurance (life and property), banking, investments, and the algebraic formulas in books about the radio.

2. Ability to perform the ordinary measurements of the household, such as length, area, volume, capacity, weight, temperature, and time, as these arise in real life situations.

3. Common sense, judgment, and self-reliance, and the habit of honesty. Mathematics is not a mechanical science; it demands the habit of using common sense in attacking a problem, and the home requires the same. The pleasure of mathematics lies to a considerable extent in the growth of self-reliance, and this same reliance is readily transferable to life problems. Mathematics is preëminently the science which requires the individual to be honest with himself, and it lies with the teacher to see that the habit of absolute accuracy and honesty is transferred to other fields.

4. The possible creation of new ideals of parenthood through the mathematical laws of heredity.

5. The cultivation of thrift in matters of time, money, and natural resources. This will include the following:

a. The development of regular habits, time schedules, and punctuality.

b. The desire to conserve natural resources, health, and energy.

c. A proper appreciation of food values, the cost of wastefulness, and the importance of thrift in providing for old age.

d. A proper sense of values through an intelligent consideration of the relative advantages of buying for cash or having charge accounts, of owning or renting, of contract purchases or mortgages, of high or low rates of interest, and in general of conservative investments or "get-rich-quick" schemes.

e. The value of estimating costs, such as that of furnishing a home, of feeding and clothing a family, of getting an education, and of travel, amusements, and the like.

Relation to a Person's Vocation. There is no vocation that does not make some use of mathematics; there are various vocations that use it extensively. The direct use in computation and measuring is self-evident. The indirect use depends largely upon the degree of transfer cultivated under the teach-

er's guidance. The following are among the mathematical objectives related indirectly to one's vocation:

1. Habits of neatness in work, accuracy of results, and uniformity of expression in similar cases. Neatness of solution in algebra may, with proper encouragement, lead to neatness of expression in other fields, possibly even in industry or trade. Accuracy of results and the habit of checking are closely bound to the same habits in commercial life.

2. Ability to interpret graphs, which has become a necessity in the great vocations, — investments, corporation finance and production, agriculture, and machine-shop practice. It extends, however, beyond mere statistics of past conditions; it has come to form a good basis for judging the future trend of demand, of financial movements, and of supplies of materials.

3. Ability to use and understand symbolic expressions. The use of the formula to represent natural laws, business conditions, agricultural information, and industrial procedure is coming to be more and more apparent. Just as music is the nearest international vocal expression of the race, so the algebraic formula is the nearest international written expression, understood in all languages in which modern mathematics is expressed and in a large number of the vocations which these languages describe.

4. Ability to analyze a complex situation into simpler parts, and then to select the essentials and reject the nonessentials. Such ability shows itself in problem solving, in geometric demonstration, and equally in the situations developed by one's vocation.

5. Ideals of perfection, whether in trigonometry or in craftsmanship, in an algebraic proof or in a legal brief. Nowhere is the habit more bound up with the subject than in mathematics; and once the ideal is established and the habit is formed, poor must be the teacher who cannot and does not see that each is transferred into possible vocational lines. Precision of thought, precision of statement, precision of result, — these are ideals in the automobile industry, in the pulpit, and in banking practice as well as in the domain in which they are first made prominent, the mathematics of the schools.

6. Some knowledge of law procedure in modern business activities, including such items as partnership and corporations, banking, savings and commercial accounts, deposit slips, pass books, checks, drafts, interest, discounts, foreign money, travelers' checks, letters of credit, and the like.

7. An understanding of the essential ideas in investments, including simple and compound interest, the different ways to invest money, and what constitutes a good investment.

Relation to Civic Education. The study of our relation to the state is, to a large degree, bound up with the study of mathematics. Some of the bonds between the two are set forth in the following list of objectives:

1. A broad conception of the relation of the individual to his local community, to the state, and to the nation. This is seen in the individual's duty to support the state through taxes, in his duty to interest himself in politics enough to influence the proper expenditures of the government, in his duty to be interested in public health so as to understand the mathematics of disease, and to be interested in the finances of public charities and general welfare. In large part, this concerns merely arithmetic; to a less extent it concerns the formulas developed in and expressed by algebra.

2. An appreciation of beautiful forms, generated in intuitive geometry and applied in city planning, the laying out of park systems, and in architecture and formal decoration. The appreciation of our noble architecture, and a mental protest against much of the geometric decoration that defaces it, — these should be and easily are cultivated in the study of mathematical forms in the junior high school.

3. The ability to relate graphs to statistical tables and to understand the significance of each.

4. The cultivation of honesty, of truthfulness, and of accuracy of statement.

5. The ability to understand such financial and commercial terms as "checking accounts," "thrift accounts," "community chests," "real estate," "tax levies," "tax rate," "assets," "liabilities," and "personal property."

6. An understanding of budgets of the city, state, and nation.

7. The meaning of foreign exchange and its significance in connection with commercial matters.

8. An appreciation of the place and significance of public utilities, government enterprises, civic improvements, and the like.

Relation to the Leisure of the Individual. The highly civilized world is coming to enjoy more leisure than was the case with preceding generations. The eight-hour day, the forty-four-hour week, and the suggestion of a five-day working week are significant of the replacing of human power by that of the machine. What has mathematics to do with a man's use of his leisure? Aside from contributing to his power to raise the money to allow him to have any leisure at all, the following objectives are worth at least a passing consideration:

1. A taste for the beauty of form as stimulated by the study of forms in intuitive geometry, particularly with respect to regular figures and to symmetry. It can hardly be doubted that this can easily be made transferable to daily life, creating an appreciation of geometric form in decoration and in architecture.

2. The habit of reading mathematical articles relating to the ever-interesting and ever-developing sciences of astronomy and physics.

3. The appreciation of mathematical forms in nature, the study of which subject opens up a new world of thought to many lovers of outdoor life.

4. The cultivation of mathematics for its own sake, whether as a pure recreation or as a science applicable in a high degree in almost every vocation or avocation of man.

5. A constructive imagination, few fields offering a richer and a greater range of possibilities than mathematics.

Relation to Ethical Character. It may seem at first sight that mathematics can claim no kinship with ethics, but there are nevertheless various interesting points of contact. Some of these are suggested by the following mathematical objectives:

1. A respect for absolute truth, without which mathematics must fail and ethics must cease to exist.

2. Habits of self-reliance and of honesty, — characteristics of both ethics and mathematics.

Relation to Religion. The points of contact between mathematics and religion are many more than the superficial mind would think possible. The most significant bonds concern phases of mathematics beyond the reach of the junior high school. A few, however, may be mentioned as follows:

1. Coming in contact with the infinite and studying the significance of the term.

2. Seeing that laws that seem perfectly evident in the finite domain cease to be valid in the regions of the infinite. For example, $\infty + 4 = \infty$, but it does not follow that we can subtract ∞ from equals and have $4 = 0$.

3. Recognizing our infinitesimal nature in the midst of the great cosmos in which we find ourselves.

4. Seeing that the imaginary becomes perfectly real in many situations.

5. Realizing that "God eternally geometrizes," as seen in geometric forms in crystals, in vegetation, and in animal life.

5. Important Concepts of Elementary Mathematics

General Purpose. The purpose of considering these concepts is to allow the teacher to make certain that the significant ones have been satisfactorily brought to the attention of the pupils. It is not the purpose to consider at this time the important abilities to be cultivated, these being presented later.

Classification. In order to aid the teacher who is specially interested in any given branch, or who, in a course in general mathematics, desires to see if the important concepts in each topic are recognized, these concepts are here classified according to special subjects. It must, however, be understood that various concepts included in one branch are also needed in one or more other branches. In all cases it is expected that the concepts shall be understood but not necessarily defined; and if defined, that the definitions will not in general be memorized.

A. *Important Concepts in Business Arithmetic*

Account, bank
Amount, of a commission
  of a debt
  net
  of a note
Annuity
Approximation
Assessment
Average
Bank
Bill
Bond
Broker
Brokerage
Budget
Calorie
Capital
Certificate, postal-savings
Checks, bank
  on operations
  travelers'
Collateral
Collector of taxes
Commercial paper
Commission
Compensation, workmen's
Corporation
Coupons
Credit, letter of
Creditor
Debtor
Deposit slip
Discount, bank
  on bills
  on a note
Dividends on stock
Draft
Drawee
Drawer
Drawing, scale
Drawing, working
Duty on imports
Face, of a bond
  of a note
  of a policy
Gain
Graphs
Income
Indorsement
Insurance
Interest, simple
  compound
Invoice
Loss
Lumber measure
Maker of a note
Margin
Maturity of a note
Money order
Mortgage
Note, at a bank
  with collateral
  demand
  interest-bearing
  joint
  negotiable
  promissory
Overhead
Par
Payee
Payments, partial
Per cent
Policy
Post office
Premium
Price, list
  marked
  selling
Principal
Proceeds

Profit
Quotation, stock
Rate, of commission
 of discount
 of income
 of insurance
 of interest
 of tax
Receipt
Revenue, customs
 internal
Share of stock
Statement, bank
 store
Stock in a corporation
Tariff
Taxes
Trade acceptance
Valuation, assessed
Value, face
 market
 par

B. Important Concepts in Intuitive Geometry. (*See also E*)

Altitude
Angle, acute
 bisector of an
 of depression
 of elevation
 obtuse
 size of an
 straight
Angles, adjacent
 alternate
 complementary
 corresponding
 equal
 interior
 made by a transversal
 supplementary
 of a triangle
 unequal
 vertical
Arc
Area
Axis of symmetry
Base, of a polygon
 of a solid
Bisector, of an angle
 of a line
 perpendicular
Center of a circle
Center, of a regular polygon
 of a sphere
Central angle
Circle
Circumference
Compasses
Complement
Cone
Congruence
Construction of a figure
Cube
Curve
Curve surface
Cylinder
Degree, angular
Diagonal of a polygon
Diagonal of a solid
Diameter
Direction
Distance
Ellipse
Equality
Figures, congruent
 geometric
 similar
 symmetric
Formula
Height of a plane figure

Height of a solid
Hemisphere
Hexagon
Hypotenuse
Length
Line, broken
  curve
  horizontal
  segment
  slanting
  straight
  vertical
Lines, equal
  oblique
  parallel
  perpendicular
  proportional
Measurement
Metric system
Midpoint
Octagon
Pantograph
Parallel
Parallelogram
Pentagon
Perimeter
Perpendicular
Pi (π)
Plane
Point
Polygon
Position
Prism
Proportion
Protractor
Pyramid
Quadrilateral
Radius, of a circle
  of a cone
  of a cylinder
  of a regular polygon
  of a sphere
Ratio
  of circumference to diameter
Rectangle
Rectangular solid
Rhombus
Root, square
  cube
Ruler
Scale drawing
Section
Sector
Semicircle
Side, of an angle
  of a polygon
Similarity
Size
  of an angle
Solid
  rectangular
Sphere
Square
Straight angle
  line
Supplement
Surface, curve
  plane
Symmetry
Transversal
Trapezoid
Triangle, acute
  equilateral
  isosceles
  obtuse
  right
Triangles, congruent
  similar
Vertex, of an angle
  of a cone
  of a polygon
  of a solid
  of a triangle
Volume

C. Important Concepts in Beginning Algebra

Abscissa
Addition
Aggregation
Axiom
Binomial
Cancellation
Checks on operations
Coefficient
Constant
Coordinates
Cube of a number
Degree
Denominator
  lowest common
Dependence
Dependent variable
Division
Elimination
Equations, checking
  of condition
  degree of
  equivalent
  fractional
  inconsistent
  indeterminate
  linear
  literal
  numerical
  quadratic
  simultaneous
Exponent, fractional
  integral
  negative
  positive
  zero
Factor, literal
  monomial
  numerical
Formula
Fraction, algebraic
Fraction, as an exponent
  proper
  signs of a
  terms of a
Fractions, clearing of
Graphs, bar
  broken-line
  circular
  curve-line
Identity
Independent variable
Index of a root
Known quantity
Members of an equation
Monomial
Multiple
  lowest common
Multiplication
Negative exponent
Numbers, algebraic
  directed
  fractional
  integral
  irrational
  negative
  positive
Numerators
Operations, with directed numbers
  signs in
Ordinate
Origin
Parentheses
Polynomial
Power
Product
Proportion
Quadratic equation
  trinomial
Quotient

Radical
Ratio
Reciprocal
Reduction of fractions
Root, cube
  of an equation
  index of a
  square
Satisfy an equation
Scale, algebraic
Sides of an equation
Signs
Solution, checking a
  of an equation
Square
Substitution
Subtraction, signs in
Symbols
Terms, of a fraction
  lowest
  of a polynomial
  of a ratio
  similar
Transposition
Trigonometry (See *D* following)
Trinomial
Unknown quantity
Value, absolute
Variable
Variation
Zero, as a number
  as an exponent

D. Important Concepts in Beginning Trigonometry

Cosine
Cotangent
Functions
Indirect measurement
Ratio
Sine
Tangent
Trigonometry

E. A Few Optional Concepts in Beginning Demonstrative Geometry in Addition to Those Already Listed under B (Intuitive Geometry) to be Given if Required in the Junior High School

Angle, central
  exterior
  inscribed
  oblique
Axiom
Chord
Converse
Corresponding parts
Demonstration
Distance, between parallel lines
  from a point to a line
Foot of a perpendicular
Inclination
Intercept
Lines, concurrent
Locus
Median
Oblique lines
Postulate
Problem
Proof, nature of a
Pythagorean Theorem
Segment
Side, included
Sides, adjacent
Solution, nature of a
Tangent
Theorem

6. Abilities in Arithmetic

General Purpose. Arithmetic is practically endless in its applications. If we wished to open up new lines of problems we might introduce the arithmetic of the machine shop, of the carpenter shop, of automobile manufacturing, of foreign exchange, of chemistry, and of hundreds of other branches. These are all technical, however; that is, each relates to some special department rather than to the needs of people in general. The question that must be considered first of all in making up a course of study relates to the kind of general information that all people need. This, therefore, is a paramount question in arithmetic as well as in other subjects.

This having been considered, the next question concerns the abilities necessary in dealing with the topics thus selected. One of the most notable changes in the teaching of arithmetic in the last quarter of a century is that which relates to the ability of pupils to do certain specified and important things in the way of computation and to solve certain types of problems. It is the purpose of the second part of this discussion to bring into prominence only the great features of arithmetic which demand specific abilities. In addition to these it is desirable but not necessary to acquire the ability to use the slide rule.

Classification. While the desirable information varies somewhat according to important local industries, the main branches are the same in all schools. They may be classified in various ways, and for an extended study of the subject the teacher is referred to other sources. For the teacher of mathematics in the junior high school, who wishes a brief statement of the subject from the standpoint of the upper grades, the following is suggested as a workable scheme. In connection with the general information demanded there will be considered the special abilities.

A. Fundamental Processes

1. *Reading numbers.* This may be limited to billions, the trillion being too rarely met with to demand attention in schools. As the pupils proceed they will perhaps find that in science they

will need much larger numbers, but that these are rarely called by name. In actual practice a scientist will write 2.35×10^{16} for the number 23,500,000,000,000,000, but he will seldom try to read the number except as 2.35 times 10 to the 16th power.

2. *Degree of accuracy in writing results.* This means that the pupil should understand the significance of the expressions "correct to hundredths," "correct to four decimal places," and "correct to five significant figures," as explained on page 82.

3. *Degree of accuracy in measurement.* This means that the pupil should understand that there is no such thing within his mental grasp as an absolutely accurate measurement, but that in certain activities of modern life a very high degree of approximation is demanded.

4. *Economy of operation.* This means that the pupil should not only know how to perform the ordinary operations with whole numbers and with both common and decimal fractions, but that he should know how to use such short cuts as are really valuable.

5. *Abilities summarized.* The abilities necessary in the computations required in ordinary business and industrial pursuits may be summarized briefly as follows:

a. Ability to perform accurately the fundamental operations with whole numbers; with decimals extending to ten-thousandths; and with fractions with denominators 2, 3, 4, 8, and 16, less commonly 5, 6, and 12, and possibly 10.

b. Ability to express a ratio as a common fraction, and as per cent or a decimal.

c. Ability to use the short cuts in multiplying by 10, 25, 50, 100, 1000, $12\frac{1}{2}$, $16\frac{2}{3}$, $33\frac{1}{3}$, and 75; and in dividing by 10, 25, 50, and 100.

d. Ability to express in decimal and per-cent form the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{6}$, and the eighths.

e. Ability to find a square root to the nearest hundredth by any convenient method, to check all operations, to estimate results in advance, and to use such tables as those of interest, square root, and cube root.

B. Per Cents

1. *Meaning of per cent.* This means that the pupil should understand that percentage is merely a part of decimal fractions; that 6% is only another way of writing 0.06 or $\frac{6}{100}$; and that if it were not for business customs we could get on very well without the symbol % and the term "per cent."

2. *Computation of profit on the cost.* If an article is bought for \$1, and the expenses of doing business are 60¢, the total cost is \$1.60. If the rate of profit is to be 20% on the total cost, the profit is 20% of \$1.60, or 32¢, and the selling price is \$1.92. This is the older method and is still used in some businesses. In many cases the word "margin" is replacing the term "gross profit." It is the difference between the cost and the selling price, and includes both the expenses of doing business and the profit.

3. *Computation of profit on the selling price.* In the preceding case, if the profit is to be 20% on the selling price, then \$1.60 must be 80% of the selling price. Therefore the selling price is \$2 and the profit is 40¢. This is a more modern method and is used by many large business houses. The growing use of "margin" for gross profit should be understood.

4. *The meaning of discount.* This refers to discount on sales and discount on notes, the mathematics being the same in each case.

5. *Ability to use the six-per-cent method.* This applies to both interest and discount on a note, the two being mathematically the same.

6. *Meaning of commission.* This refers to commission on sales of produce, land, or other property.

7. *Significance of the simple-interest formulas.* This means the formulas $i = prt$ and $A = p(1 + rt)$.

8. *Abilities summarized.* The abilities necessary in the computations involving per cents may be summarized as follows:

a. Ability to find any required per cent of a number, the net price of an article on which a discount is given, what per cent one number is of another, a number of which a certain per cent is given, and per cents of increase or decrease.

b. Ability to use fractional equivalents of $12\frac{1}{2}\%$, $37\frac{1}{2}\%$, 25%, 50%, 75%, and $33\frac{1}{3}\%$. Cases involving $87\frac{1}{2}\%$, $66\frac{2}{3}\%$, and certain others required in various courses of study are not of so much practical importance.

c. Ability to understand the meaning of such expressions as $2\frac{1}{2}\%$, 3.5%, 5%, 0.5%, 100%, 200%, 125%, and 12.5%.

C. Problems of the Home

1. *Personal accounts.* This means the knowledge of and ability to keep personal accounts. The account should show all receipts, all expenses, and the balance at the end of each week or month.

2. *Household accounts.* This means the same as the preceding, only applied to the household. The accounts are such as a housewife might keep.

3. *Budget system.* This means the knowledge of how to prepare a budget for an individual or a household, showing probable receipts and expenses for a year. In this way we are led to plan to save a certain amount each year so as to prepare for the "rainy days" of life.

4. *Personal inventory.* This means that at the close of each year all adults should "take stock," writing down the amount of property that each one owns. This will encourage saving a reasonable sum each year.

5. *Typical home problems.* This means that the pupil should have a reasonable amount of information relating to food values, food costs, clothing values, clothing costs, discount sales, heating, lighting, decorating, sewing, repairs, and the reading of a gas, electric, or water meter.

The arithmetic abilities needed for such problems have been summarized under *A* and *B* above.

D. Abilities respecting Measurements

1. *Counting.* This means that the pupil should know the meaning of such terms as dozen, score, and gross. These terms are old but are still used, although the score is not very common and the gross is used chiefly in wholesale purchases.

2. *Length.* This means particularly the common units of inch, foot, yard, and mile, with intimate knowledge of the rod in rural communities. It is interesting to mention the "light year," used in astronomy, and meaning the distance that light travels in one year. There are stars that are hundreds, thousands, and possibly millions of light years away.

3. *Weight.* This means a knowledge of the ounce, pound, and ton. The long ton is not of great importance in these grades except in mining communities and a few special localities.

4. *Liquid units.* These are the pint, quart, and gallon. The gill is becoming obsolete except in the buying of cream in certain cities.

5. *Dry units.* These are the quart, peck, and bushel, with some mention of the pint.

6. *Square units.* These are the squares of the units in No. 2, together with the acre and, in rural communities, the square rod.

7. *Units of time.* These are the second, minute, hour, day, week, month, year, decade, and century.

8. *Angular units.* These are the second, minute, degree, right angle, straight angle (180°), and circumference (360°). Decimal parts of the degree are replacing minutes and seconds.

9. *Kitchen units.* Among these are the teaspoonful, tablespoonful, and cupful. A pint of water weighs approximately a pound.

10. *Foreign money.* It is desirable to know the value in our money of such foreign units of value as are substantially fixed. These are approximately as follows: £1 English = \$4.87, 1 German gold mark = 24¢, 1 Japanese yen = 50¢, \$1 Mexican = 50¢, 1 Swiss franc = 1 Spanish peseta = 19.3¢. The subject has little importance to the great mass of our people under the disturbed conditions of foreign exchange, and is usually omitted at present.

11. *Metric system.* The chief units needed are the meter, kilometer, centimeter, millimeter; the squares of these measures; the cubic meter, cubic centimeter, and cubic millimeter; the liter; and the gram and kilogram, with a knowledge of the significance of the prefixes "centi-" and "milli-" in centigram and milligram.

12. *Convenient equivalents.* While not to be memorized, it is desirable to know of the following approximate relations: 1 bu. contains $1\frac{1}{4}$ cu. ft.; 1 gal. contains 231 cu. in.; 1 cu. ft. of water weighs $62\frac{1}{2}$ lb.; 1 cu. ft. of water contains $7\frac{1}{2}$ gal.; 1 T. of hay occupies 500 cu. ft.; 1 T. of hard coal occupies 35 cu. ft.; 1 T. of soft coal occupies 42 cu. ft.

13. *Abilities summarized.* In order to use the above information the pupil must have the following abilities:

a. To know and use the common tables of measure, and to know and use the few important units of the metric system.

b. To add or subtract numbers like 6 ft. 4 in. and 8 ft. 11 in., the work being generally limited to feet and inches, yards and inches, or pounds and ounces. The ability to find the difference between dates is also useful, especially in cases where interest has to be computed for years, months, and days.

c. Ability to multiply or to divide in such simple cases as that of 8 ft. 6 in. by 5.

d. Ability to perform such simple reductions as that of feet or yards to inches and the reverse, and of pounds to ounces or tons and the reverse.

e. Ability actually to measure distances, areas, volumes, and weights to a reasonable and stated degree of accuracy.

E. Concerning the Store and the Shop

1. *Bills.* The general nature and meaning of a bill for goods.

2. *Making change.* The common method of making change; additive subtraction.

3. *Cash registers.* Their nature and purpose.

4. *Inventory.* Necessity for taking it once or twice a year.

5. *Invoice.* Its nature as compared with a store bill. Discount or discounts allowed.

6. *Bill of lading.* Its purpose and general nature.

7. *Contract.* The general nature of a contract.

8. *Parcel post and express.* How to ship goods in small quantities.

9. *Transportation problems.* How to ship goods in large quantities.

10. *Commission on sales.* Although already considered under per cents, the subject is also of importance in this connection.

11. *Overhead.* Meaning of "overhead," or the cost of doing business, and the necessity of adding this to the net cost in computing the total cost of goods.

12. *Elements of a pay roll.* How a pay roll is made out and how the money is drawn from the bank to allow for the precise change in making payments.

13. *Abilities summarized.* The arithmetic abilities required in connection with the above items may be summarized as follows:

a. To make out and understand bills, inventories, invoices, and a pay roll, each in its simplest form. It is not necessary to be able to make out a bill of lading or a contract, this being too technical for the schools.

b. To check bills for goods.

c. To find discounts on bills and invoices, such as "15% off" or "$\frac{1}{6}$ off."

d. To understand in a general way the use of a cash register and to know that this is only one of various types of machines for computing.

F. Concerning Banking

1. *Post-office banks.* Significance of the Postal Savings System, which is in effect a government savings bank.

2. *Savings banks.* Their purpose; how business is done with such banks; the meaning of compound interest.

3. *Commercial banks.* How money is deposited; how it is withdrawn: pass book, checks, drafts; borrowing money; collateral.

4. *Transmitting money or its equivalent.* Checks, drafts, money orders. Recognition that the term "draft" is rapidly being replaced by "check" in the case of bank drafts (checks).

5. *Trade acceptances.* A new form of draft accompanying a shipment of goods. Only a general idea of these acceptances is necessary.

6. *Exchange.* The cause of the great fluctuation in foreign exchange in recent years. The reason why the subject of foreign exchange is not at present desirable for general school

instruction, except in connection with countries in which the money is on a gold basis, like ours.

7. *Building and loan associations.* In localities where these have been successful through a series of years and are conservatively managed, their nature should be explained. If possible this should be done by an official of the association.

8. *Abilities summarized.* The arithmetic abilities with respect to the above may be summarized as follows:

a. As in all such cases, the ability to compute with absolute accuracy.

b. To find the discount on a note (1) when no interest is specified and (2) when it is interest-bearing.

c. To balance a bank statement of deposits (credits) and withdrawals (debits) and to verify it by the stubs in a check book.

d. To make out a deposit slip for a bank.

e. To write and indorse checks and to keep the record on the stubs of a check book accurately.

G. Community Arithmetic

1. *Benefits from the community.* This means that the schools should make clear the benefits which everyone receives from the nation, the state, the city, the village, and the community in general. These include free schools, free roads, parks, water supply, street cleaning, street lighting, sewerage, and the like.

2. *Our duty to the community.* This means that the schools should lead the pupils to take an interest in the general nature of the expenses of all the above branches of community government, should show the justice of the levying of taxes, and should show our duty in paying such taxes. After all, it is our government, and we should put the best men in office and should pay what the government needs, this government including our local community as a part.

3. *Typical community interests.* This means that the pupils should be informed concerning the cost of good roads, of public buildings, of street lighting, of water supply, and the like, and that they should then solve a reasonable number of local problems relating to this work.

4. *Insurance.* This means that the social benefits of insurance should be made clear, many joining to help those who are in need through ill health, accident, fire, or the loss of members of the family. The three or four leading types of life-insurance policies should be understood; the nature of fire insurance should be explained; the workmen's compensation laws should be discussed, and accident and casualty insurance should receive attention. The pupils should understand the meaning of a policy, of the rate (premium), and the face.

5. *Abilities summarized.* Arithmetically, the above work requires no ability except to compute accurately. It is not expected that the schools should attempt to teach the theory of insurance or the method of community borrowing. The topics mentioned should be looked upon as informational rather than mathematical.

H. Thrift and Investments

1. *Savings.* This means some attention to the importance of saving in a country where a large majority of our people are dependent upon others when they are sixty years old. Everyone should save while he is able. A bank account started early and maintained is likely to grow year by year.

2. *Safe investments.* It is the duty of the schools to warn all pupils against reckless investments. The difference in income between 5% and 8% is only $3 on a hundred per year, but the former is likely to be absolutely safe while the latter is not. It is desirable that all should realize the danger of gambling in stocks.

3. *Difference between stocks and bonds.* People with even relatively small amounts of money to invest are in the habit at present of buying stock in some large corporation, like a railroad, or of buying a bond of some such corporation or of a city, a state, or the national government. The distinction between stocks and bonds should therefore be understood, and their relative desirability should be explained.

4. *Information about stocks.* The pupils should understand such terms as preferred stock, common stock, certificate of

stock, dividends, par value, above and below par, brokerage, and stock exchange.

5. *Information about corporations.* The pupils should understand the general nature of a corporation, the method of electing directors and officers, and the way in which corporations are financed and managed.

6. *Investments in bonds and mortgages.* The difference between a bond and a mortgage should be understood, and that the latter is security for the former. The question of the desirability of a bond and mortgage as an investment should be considered.

7. *Abilities summarized.* As in the case of community arithmetic, the above work is chiefly informational. The computations involved are generally the elementary operations and the use of per cents. Specifically, the following abilities should be considered:

a. To write a promissory note.

b. To find the date of maturity of a note.

c. To find the time between the date of a note and the date of maturity.

d. To find the interest on a note.

e. To compute the dividends on stocks, given the par value and rate.

f. To find the rate of income, given the principal invested and the amount paid.

g. To distinguish between stocks and bonds as forms of investment.

h. To understand the meaning of newspaper quotations of the prices of stocks and bonds.

I. Statistics and Graphs

1. *Relative value of a statistical table and a graph.* This means that each has a definite place in business life: that one is generally more precise in details while the other is more vivid in presentation.

2. *Types of graphs.* The pupil should be familiar with the bar graph, curve-line graph, circular graph, and pictorial graphs of various kinds, such as are seen in current journals and magazines.

3. *Interpretation of graphs.* As already stated, this is more important than the drawing of graphs. It means that the story which the graph tells should be clearly understood.

4. *Abilities summarized.* The abilities required with respect to this topic are to

a. Make such statistical tables as those relating to the height or weight of pupils, to school attendance, and to weather reports.

b. Make and read such tables as those referring to height in relation to age, or weight in relation to either age or height.

c. Draw, with or without squared paper, graphs in the form of bar diagrams, broken lines, or curve lines. The circular graph is too difficult for construction by the pupils.

d. Find the average in a group of numbers.

e. Interpret graphs of the various kinds above mentioned, and also circular graphs and pictorial graphs such as are found in magazines and newspapers.

f. Criticize graphs that give a false impression of the statistics represented.

g. Decide upon the best type of graph for a given set of statistics.

h. Locate points with respect to two perpendicular axes OX and OY, using x for the abscissa and y for the ordinate.

i. Compare two or more statistical graphs drawn to the same scale on a single pair of axes.

j. Understand and use directed numbers in connection with graphs.

k. Select a proper scale for a graph.

J. Types of Special Projects Sometimes Suggested for Study

1. Transportation
2. Forestry
3. Automobiles
4. Gardening
5. Farm tractors
6. Pottery
7. Room decorating
8. Poultry raising
9. Machinist's work
10. Iron industry
11. Paving
12. Dairy farm
13. Plastering
14. Silo capacity
15. Papering
16. Mixing concrete
17. Contents of haystacks
18. Spraying

It should be clearly understood, however, that the "project" in general represents a very artificial type of problem. It may seem real to the teacher but be very unreal to the pupil. A girl of twelve is not likely to be very much interested in paving, nor a city boy in the contents of haystacks. A "project" is of value only when each pupil has a genuine interest in it, and even then it needs to be weighed carefully before it is allowed to take any time from the work in computation. The problems that we are called upon to solve in daily life are nearly always isolated ones or those relating to our personal, bank, or household accounts; only in very rare cases are they of the so-called project type.

7. Abilities in Geometry

General Purpose. Whether or not there is any demonstrative geometry taught in the junior high school, a considerable range of desirable information will naturally be secured. For example, pupils will come to know the simple facts of congruence by intuition, and similarly those relating to the shapes and sizes of figures and to position. These facts may, therefore, properly be considered as desirable information easily within the reach of pupils of these grades. Even if only a brief allowance of time can be given to a few simple demonstrations and exercises, the chief objective will be attained; that is, the pupil will obtain a fair idea of what is meant by a mathematical demonstration. Nevertheless it is true that geometry is such an extensive branch of mathematics that it is necessary to limit its scope in the junior high school largely to the intuitive phase of the subject. It is now proposed to set forth certain of the most important topics to be included and abilities to be developed within the boundaries thus determined.

Classification. For this purpose we shall first consider the general range of information to be expected, and next the general abilities which we can reasonably hope to develop. We shall then take up the abilities that relate more particularly to certain important details. The following classification represents the features which the teacher will wish to emphasize.

A. Simple Instruments

1. *Meaning of intuitive geometry.* As a matter of general information the pupil should understand clearly that intuitive geometry means geometry without scientific demonstrations and that it permits of the use of any drawing instruments we wish.

2. *Leading concepts.* Naturally, the leading concepts of intuitive geometry are points, lines, surfaces, and solids. We use instruments for drawing lines, either straight or curve, and we use lines for locating points. We do not use instruments for establishing surfaces or solids, except as this is done indirectly by drawing lines.

3. *Primary instruments.* The instruments primarily used are the ruler (or straightedge) and compasses. These are sufficient for elementary geometry.

4. *Additional instruments.* For convenience, in intuitive geometry, we supplement the primary instruments by others. For example, we use a T-square in drawing parallels, a protractor for drawing and measuring angles, a plumb line for establishing a vertical, a level for establishing a horizontal, a compass for measuring horizontal angles, and a transit for measuring both horizontal and vertical angles. With all these instruments the pupil should be acquainted, at least through pictures. The subject of homemade instruments will be considered in Chapter XII.

B. Shapes of Figures

1. *Regularity of figures.* This means a knowledge of what is necessary to constitute a regular plane or solid figure.

2. *Symmetry of figures.* This means a knowledge of symmetry with respect to a point, a line, or a plane.

3. *Similarity of figures.* This means a knowledge of the similarity of plane figures and of solids.

4. *The standard types of figures.* This means a knowledge of the various kinds of triangles, of quadrilaterals, of prisms, and the like. In the junior high school it does not include the oblique cone, prismoid, or other types not commonly seen by pupils.

C. *Size of Figures*

1. *Methods of measuring lengths.* This means the common methods of determining lengths and of checking the work. It includes both direct and indirect measurements; that is, measurements by means of tape lines, rulers, compasses, and other scales, and also by trigonometry.

2. *Methods of measuring areas.* This means the common methods of dividing the figure into rectangles or triangles, or approximately so; the method of drawing to scale and then measuring the area of the drawing; the finding of a volume by measuring the amount of water necessary to fill an irregular container, or the amount of water that will run over when an irregular solid is slowly lowered into a full container.

D. *Position*

1. *Point referred to axes.* This means the ability to locate a point with reference to two axes, just as we locate a place by latitude and longitude.

2. *Point referred to angle and distance.* This means the ability to locate a point in a certain direction and at a certain distance from a known point.

3. *Point referred to two points.* This means the ability to locate a point when its distances from two fixed points are known. Several possible solutions are to be considered in Nos. 3–5.

4. *Point referred to a line and a point.* This means the ability to locate a point when its distances from a fixed line and a fixed point are known.

5. *Point referred to distances from planes and points.* This means that loci with respect to planes and spherical surfaces are to be considered informally.

E. *Congruence Theorems*

1. *First congruence theorem.* This refers to the case of two sides and the included angle. All such cases may be taken with or without proof, according to the ability of the pupil or the class.

2. *Second congruence theorem.* This refers to the case of two angles and the included side.

3. *Third congruence theorem.* This refers to the case of the three sides.

F. Theorems on Angles and Triangles

1. *Equality of vertical angles.* This refers to the vertical angles formed when two straight lines intersect. It may be taken as a postulate if desired.

2. *Isosceles triangle.* This refers to the equality of the base angles of an isosceles triangle.

3. *The converse of No. 2.* This refers to the equality of the sides opposite two equal angles of a triangle. Whether or not the preceding case (No. 2) is proved, this may properly be taken at first as a postulate.

4. *Angle sum.* This refers to the fact that the sum of the angles of any triangle is 180°. It should be extended to figures of four, five, and six sides and should include the cases of both regular and irregular figures.

G. Parallels and Parallelograms

1. *Condition of parallelism.* This refers to the case of equal corresponding angles or alternate angles formed by a transversal cutting two lines, the lines then being parallel.

2. *Converse of No. 1.* This means that if the lines are given parallel, the corresponding angles or the alternate angles will be equal.

3. *Opposite angles of a parallelogram.* This refers to their equality.

4. *Consecutive angles of a parallelogram.* This refers to their being supplementary.

5. *Opposite sides of a parallelogram.* This refers to their equality.

H. Similarity

1. *First condition of similarity of triangles.* This refers to the case in which two angles of one triangle are respectively equal to two angles of another.

2. *Second condition of similarity of triangles.* This refers to the case of one angle and the proportionality of the including sides.

3. *Results of similarity.* This refers to the fact that the angles are respectively equal and the sides are respectively proportional.

4. *Pythagorean Theorem.* This may, as heretofore suggested, be easily proved by similarity.

I. General Abilities

Ability to do each of the following:

1. Use the common measures of length, area, and volume.

2. Use the metric measures of length, limited to the kilometer, meter, centimeter, and millimeter; area, limited to the squares of the units of length; and volume, limited to the cubic meter, cubic centimeter, and cubic millimeter. The decimeter will naturally be mentioned, but as a unit of linear measure it is not so important as the others.

3. Read a line segment lettered in either of two convenient ways.

4. Read an angle lettered in any one of three convenient ways.

5. Measure a line segment with a ruler, the result being accurate to the nearest tenth or sixteenth of an inch, or to the nearest millimeter, depending upon which scale is used.

6. Measure a line segment by using dividers (compasses) to transfer its length to a ruler.

7. Find approximate distances on the floor or out of doors by means of pacing.

8. Use squared paper for the purpose of finding the length of line segments transferred to it by the dividers (compasses), and the area of plane figures drawn upon it to scale.

9. Add one line segment to or subtract it from another.

10. Measure an angle by means of a protractor.

11. Given two similar figures, use proportion to compute any side when sufficient data are known.

12. Measure the height of an object by shadow reckoning, using proportion.

13. Measure the distance to an inaccessible object by means of a scale drawing or else by proportion.

14. Locate a place by using a horizontal and a vertical axis, as in longitude and latitude.

J. Drawing and Constructing

Ability to do each of the following:

1. Distinguish between drawing (either freehand or with the ruler and protractor) and construction (with only the ruler and compasses).
2. Understand a scale drawing or a simple plan of one or more rooms of a building.
3. Understand the meaning of a map drawn to scale.

Ability to draw the following figures, using the ruler, protractor, and, if convenient, a draftsman's triangle:

4. A line segment of given length.
5. An angle of a given number of degrees.
6. A right angle.
7. A line parallel to a given line.
8. A square with the sides of a given length.
9. A rectangle of any convenient size.

Ability to perform the following constructions with a ruler and a pair of compasses:

10. Construct a circle with a given radius.
11. Bisect a given arc.
12. Bisect a given angle.
13. Bisect a given line.
14. Construct a right angle.
15. At a given point on a line construct a perpendicular to the line.
16. From a given point outside a line construct a perpendicular to the line.
17. Divide a given line segment into a given number of equal parts.

Ability to construct the following figures, using only a ruler and a pair of compasses:

18. An equilateral triangle having a given side.
19. An isosceles triangle having the base and one of the equal sides given.
20. An angle equal to a given angle.
21. A line parallel to a given line.
22. An angle equal to the sum of two given angles or to the difference between two given angles.
23. A regular hexagon inscribed in a circle.
24. A square inscribed in a circle.
25. An equilateral triangle inscribed in a circle.
26. A triangle having two sides and the included angle given.
27. A triangle having two angles and the included side given.
28. A triangle having the three sides given.
29. A square having its side given.
30. A rectangle having two adjacent sides given.
31. The center of the circle of which an arc is given.
32. Angles of 30°, 45°, and 60°.
33. The perpendicular bisectors of the sides of a triangle.
34. The bisectors of the angles of a triangle.
35. The perpendiculars from the vertices of a triangle to the opposite sides.
36. The medians of a triangle.
37. A copy of a given geometric design.

K. Making Correct Inferences

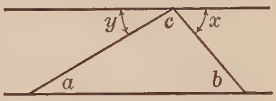
Ability to make correct geometric inferences with respect to such simple cases as the following:

1. The congruence of triangles.
2. The alternate angles formed by a transversal cutting two parallel lines.
3. The sum of the interior angles of a triangle.
4. The parallelism of two perpendiculars to the same line.
5. The similarity of triangles having the three angles of one respectively equal to the three angles of the other.

L. Proving Geometric Statements

Based upon the inferences made above, under K, or upon simple proofs based upon the axioms and postulates of elementary geometry, to prove a few propositions leading to some celebrated theorem, the following being a sample of the sequence to be used:

1. The sum of the angles of any triangle is 180°. Based upon the assumption that alternate angles are equal. Hence $a = y$, $b = x$, and so $a + b + c = x + c + y = 180°$.



2. The Pythagorean Theorem. Based upon these steps:

$$\angle C = 90° \text{ (given)},$$

the angles at P are 90°,

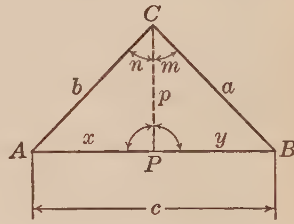
$$\angle B + \angle m = 90°,$$

$$\angle B + \angle A = 90°,$$

hence $\angle B + \angle m = \angle B + \angle A$

and $\angle m = \angle A.$

Similarly $\angle n = \angle B.$



Therefore the three angles of $\triangle APC$ are respectively equal to the three angles of $\triangle CPB$, and to those of $\triangle ACB$. Therefore all three triangles are similar. Therefore

$$\frac{c}{a} = \frac{a}{y}, \quad \text{whence} \quad a^2 = cy;$$

and

$$\frac{c}{b} = \frac{b}{x}, \quad \text{whence} \quad b^2 = cx.$$

Adding, $a^2 + b^2 = cy + cx = c(y + x) = c^2$. That is, in any right triangle

$$a^2 + b^2 = c^2.$$

The preliminary work together with the proof will require about a week, but at the end the pupils will have the satisfaction of having proved one of the great theorems of the world, and they will have an idea of the meaning of geometric proof.

These two propositions are merely typical. Others, with other sequences, may be used.

8. Abilities in Algebra

General Purpose. It should be reiterated that the great objective in elementary algebra is the ability to use formulas. This means that pupils should be able (1) to evaluate a formula and (2) to derive one formula from another. The second of these is somewhat less clearly (for the pupil) stated as the ability to "change the subject" of the formula. The pupil should understand that a formula is a shorthand statement of a rule. He should also see that a simple formula such as he uses may be represented by a graph, realizing that a rule is a translation of a formula, and a graph is its picture. The rest of elementary algebra is an elaboration of these principles. It includes equations (useful in deriving one formula from another), operations with algebraic expressions (useful in solving equations), and other features set forth in the following classification.

Classification. The classification of these abilities is based upon this modern view rather than upon the older one of beginning with operations of doubtful value and for which, at the time, the pupil could see no use.

A. Formulas

1. *Ability to discover certain rules and to translate these into formulas.* For example, the pupil should be able to discover that $a^2a^3 = a^5$, to infer the rule, and then to write the formula

$$a^m a^n = a^{m+n}.$$

2. *Ability to translate formulas into rules.* For example, he should be able to read the rule from the formula

$$(a + b)^2 = a^2 + 2\,ab + b^2$$

and to apply this to finding the square of a number like 72.

3. *Ability to evaluate formulas.* This means that, for example, given the formula $$C = 2\,\pi r,$$

he should be able to find the value of C when $r = 15$, using $\frac{22}{7}$ as the value of π.

4. *Ability to derive one or more formulas from a given formula.* This means that, for example, from the interest formula $i = prt$, the pupil should be able to derive formulas for p, r, and t.

5. *Ability to represent by a graph any simple formula.* This means that the pupil should be able to make out a table of values of the letters in a formula and to represent the formula by a curve based upon these values. The temperature formula $F = \frac{9}{5}C + 32$ represents the limit of difficulty for work of this kind.

6. *Ability to understand the dependence of one quantity upon another.* This means that in the formula just given F depends upon C for its value. Every formula represents some such dependence; in fact, all our acts, our thoughts, our successes, our failures, and our standing in school depend upon something; nothing ever happens in the lives of people without the influence of something else.

7. *Ability to work with ordinary simple formulas.* This means the ability to use formulas of the following types:

Interest	$i = prt$
Amount at simple interest	$A = p + prt = p(1 + rt)$
Amount at compound interest	$A = p(1 + r)^n$

B. The Graph of a Mathematical Law

1. *Ability to understand and use directed numbers.* These are needed in the study of graphs.

2. *Ability to represent points by means of the usual coordinate system.* This includes the appreciation of the relation of a point to a number pair (x, y), an understanding of the four quadrants formed by the axes, and the use of a table of corresponding values for the variables. For example, in the equation $x - y + 2 = 0$,

If $x =$	-2	-1	0	1	2
then $y =$	0	1	2	3	4

3. *Ability to understand the types $y = kx$ and $xy = k$.* This means that the pupil must understand the direct variation of y as x varies in the equation $y = kx$, k being some constant; and the inverse variation, as in $xy = k$. He must be able to represent

each case graphically, and must see that functional relations are better visualized by the graph than by the equation.

4. *Ability to compare graphs drawn with respect to the same axes.* This means that if we draw two related graphs on the same sheet, we can thus better compare the laws which they represent than by means of the equations.

5. *Ability to interpolate and extrapolate.* This means that by the graph we can usually approximate a value between two other values (interpolation) and we can predict the approximate values when the curve is slightly extended (extrapolation).

6. *Ability to interpret the graph of a simple formula.* This means that the pupil should be able to construct the graph of any simple formula, and then to interpret it. In every case we draw the graph for only special values of all but two variables. For example, if $A = lw$, we give some special value, say 2, to one of the letters, say w. We then have $A = 2\,l$, and we then proceed to draw the graph.

7. *Ability to interpret intersections with the axis.* This means that, for example, in the equation $y = x^2 - 3\,x + 2$ the curve will cut the x axis at the points 1 and 2; that is, if y is 0, we have $x^2 - 3\,x + 2 = 0$, and the values of x are then 1 and 2. We interpret these intersections, therefore, as the values of x in this equation. In the case of the equation $y = x^2 - 3\,x + 9$ the curve will not intersect the x axis; that is, the equation $x^2 - 3\,x + 9 = 0$ has no real roots.

8. *Ability to understand the relations between the variables in certain special cases.* This means that the graphs should show this relationship in such equations as

$$y = x^2, \quad y = x^3, \quad y = x^4, \quad \text{and} \quad y = 2\,x.$$

9. *Ability to read a compound-interest graph.* The formula is $A = p(1 + r)^n$, and we draw the graph for certain special cases; for example, that in which $p = 1$ and $r = 0.06$. We then have $A = 1.06^n$, which brings the formula under No. 8.

10. *Ability to use the graph of $y = x^2$.* This means that the pupil should construct the graph and then actually use it for finding approximate square roots like $\sqrt{1.5}$, and for finding squares.

11. *Ability to use the graph of an equation like* $y = 3x + 1$. This means that the pupil should construct the graph and see that when $y = 0$ the value of x is found where the line cuts the x axis. He should see that this is a special case of the general type $y = ax + b$. As an example of the importance of this work, let the pupils construct a graph of the temperature formula which shows the relation of the Fahrenheit to the centigrade thermometer, $F = \frac{9}{5}C + 32$, or $F = 1.8\,C + 32$.

12. *Ability to use and to interpret the graph of an equation like* $y = 2x^2 + x - 10$. This means that the pupil should construct the graph and see that it cuts the *x axis* in the points 2 and $-2\frac{1}{2}$; that is, that the roots of the equation $2x^2 + x - 10 = 0$ are 2 and $-2\frac{1}{2}$. It should be observed that this is a special case of the type

$$y = ax^2 + bx + c,$$

the latter representing a family of equations formed by giving to a, b, and c any values we choose.

13. *Ability to discover a maximum or a minimum in a graph.* This means that we can find the greatest or the least value of y in the equation of the preceding paragraph by examining the curve and locating its highest or its lowest point. This leads to many interesting problems found in the calculus.

14. *Ability to read values from a graph.* This means that the pupil should acquire the ability to read quickly and either precisely or to the required degree of approximation the values of x corresponding to assigned values of y, and also the values of y corresponding to assigned values of x.

15. *Ability to use graphs in allied fields of science.* This means that the pupil should develop the ability to make use not only of the formulas in intuitive geometry but of the simpler ones in physics as given in any standard text, of the simpler ones in business, and of such other formulas as he may see in reading about radio or in his work in elementary science.

16. *Ability to distinguish between the significance of a graph of statistics and that of a mathematical law.* For example, a statistical graph shows a probable tendency, while the graph of a mathematical law shows a certain one.

17. *Ability to express a ratio graphically.* This means that given $\frac{x}{y} = 4$, we can write the equation $x = 4\,y$ and can then construct the graph. From this graph we can find the value of x that goes with any special value of y, and vice versa. In the same way we can study the graph of the general case $\frac{x}{y} = k$, as already stated in No. 3 above.

C. Linear Equations in One Unknown

1. *Ability to translate a verbal problem into an equation.* This involves two somewhat distinct abilities illustrated by the following problems:

a. What is the cost of 8 pencils at k cents each? The answer is expressed by the equation $C = 8\,k$.

b. What is the number which when doubled and then increased by 5 is equal to 12? The equation is $2\,n + 5 = 12$.

2. *Ability to translate an equation into words.* This is the reverse of the preceding objective. It means that equation $x^2 + 1 = 10$, for example, may be translated as follows: Find a number whose square increased by 1 is 10.

3. *Ability to solve equations of the type $ax + b = c$.* This means that the pupil should be able to solve any linear equation of the form $2\,x + 7 = 19$. It involves the ability to use each of the first four axioms if necessary.

4. *Ability to understand the significance of the graph of an equation of the type $y = ax + b$.* This means that such a graph may be used for solving an equation like $3\,x - 7 = 8$, this being one of the family of equations of the type mentioned.

5. *Ability to solve linear equations containing common or decimal fractions.* The types of equations selected should be those relating to the necessary formulas of mensuration, science, or business.

6. *Ability to use equations in solving problems.* As far as possible, the problems should be such as relate to elementary science or to simple business conditions. Since the technicalities of science and business, however, are often beyond the experiences of the pupils, problems of a similar type but of artificial content must often be used.

7. *Ability to interpret artificial numbers in a result.* This means that such artificial numbers as fractions, certain roots, and negative numbers sometimes have a meaning in the solution of a problem and sometimes do not. For example, consider this problem: The number of times I picked up a pencil today is the value of x in the equation $2x+3=2$. What is the number? The result is that $x=-\frac{1}{2}$, which means nothing in the problem: (1) because I cannot pick it up half a time, and (2) because I cannot pick it up a negative number of times. The interpretation therefore is that the problem, as stated, has no solution.

D. Directed Numbers

1. *Ability to represent directed numbers graphically.* This means that the pupils should be able to construct an algebraic scale upon which they may represent all types of numbers known to them, — positive, negative, integral, fractional, and surd (that is, roots like $\sqrt{2}$), thus extending the general notion of number. The "imaginary number," like $\sqrt{-1}$, does not form a part of this system.

2. *Ability to use directed numbers practically.* This means that in practical problems such numbers as $-\$1000$, $-\frac{1}{2}$ in., and -75 lb. usually have a meaning, and that this meaning should be recognized and understood.

3. *Ability to add or to subtract directed numbers.* In either case the numbers may be written either in a column or in a row, some of the numbers being positive and some negative, or all being of the same kind.

4. *Ability to multiply by a positive or by a negative number.* This means that the pupil should develop the ability to perform such multiplications as $2\frac{1}{2}\times 6$, $3\times(-7)$, $(-2)\times 3$, and $(-3\frac{1}{2})\times(-2\frac{3}{4})$, the numbers being arranged either in a column or in a row. It carries with it the inverse order, illustrated by the cases of $(-2)\times 3$ and $3\times(-2)$.

5. *Ability to divide by a positive or by a negative number.* This operation, taught as the inverse of No. 4, includes the cases $4\div\frac{1}{2}$, $6\div(-2)$, and $(-6)\div(-2)$, together with the inverse case $(-6)\div 2$.

6. *Ability to use directed numbers in graphs.* This has already been mentioned in connection with graphs (pages 58 and 68).

7. *Ability to use directed numbers in formulas.* This means that such ideas as negative weight, negative direction, and negative pressure now enter into many practical formulas and their significance must be understood.

8. *Ability to remove either one or two sets of parentheses.* This means especially the case in which negative numbers are involved or in which a negative sign precedes the parentheses.

9. *Understanding of the double use of the signs + and −.* This means that the pupil should recognize that these signs are used both as signs of quality (as when we speak of a positive or a negative number, say $+2$ or -7), and as signs of operation (as in the case of $9+3$ or $9-5$).

E. Operations with Polynomials

1. *Addition, subtraction, multiplication, and division.* This means that these four operations, known as "the fundamental operations," are to be performed with polynomials in the manner described under directed numbers. It does not mean, however, that much attention will be paid to this kind of work, in itself quite unimportant, except when it contains easy polynomials such as will actually be needed in some type of applied problem.

2. *Multiplication of a binomial by a monomial.* This represents the most important type of multiplication beyond the case of monomials only. It should be effected with or without using parentheses, and should be illustrated geometrically by the use of a rectangle.

3. *Multiplication of any polynomial by a monomial.* This is much less important in practical work than is the preceding case.

4. *Multiplication of a binomial by a binomial.* This represents the most important type of multiplication of a polynomial by a polynomial. It should be illustrated geometrically in certain instances by the use of a rectangle of which the base is $a+b$ and

the height is $x + y$. This should lead to the development of these formulas:

$$(1)\ (a + b)(x + y) = ax + ay + bx + by$$
$$(2)\ (x + a)(x + b) = x^2 + (a + b)x + ab$$
$$(3)\ (a + x)(a + x) = (a + x)^2 = a^2 + 2\,ax + x^2$$
$$(4)\ (a + x)(a - x) = a^2 - x^2$$

with their variants in cases like $(x + a)(x - b)$, $(x - a)(x - b)$, $(a - x)^2$, and $(ax + b)(cx + d)$.

5. *Division of a binomial by a monomial.* This should be looked upon as the inverse of No. 2 and should include both integral and fractional forms.

6. *Division of a polynomial by a monomial.* This is of less practical value than the preceding case, but should be treated in the same way.

7. *Division of a polynomial by a binomial.* This is of value, at present, only in leading up to a case in factoring that is sometimes helpful in solving equations. The limit of difficulty should be a case like $(x^2 - x - 12) \div (x - 4)$.

8. *Using symbols of aggregation.* This should be limited to cases that are needed in connection with simple formulas.

9. *Understanding the value of factoring.* This means that the pupil should recognize the two chief values of factoring — a subject that often seems valueless. These are (1) that it enables us to transform one formula into another which can be more easily evaluated, and (2) that it can sometimes be used to advantage in solving equations. For example, if $A = \pi R^2 - \pi r^2$, this is more easily evaluated for $R = 7.8$ and $r = 2.2$ by writing it $A = \pi(R + r)(R - r)$. Furthermore, we can easily solve the equation $x^2 + 4\,x - 21 = 0$, when we see that $(x - 3)(x + 7) = 0$. However, this is not so important as the first value.

10. *Understanding factoring as the inverse of multiplication.* For example, because $(2\,x + 3)(5\,x + 2) = 10\,x^2 + 19\,x + 6$, we see that the factors of $10\,x^2 + 19\,x + 6$ are $2\,x + 3$ and $5\,x + 2$.

11. *Ability to remove a monomial factor.* For example, to see that $ax^2 + bx = x(ax + b)$. This is helpful in the evaluation of a formula or in deriving one formula from another.

12. *Ability to factor the general quadratic trinomial.* This means the factoring of expressions in the form $ax^2 + bx + c$. For example, $2x^2 - 11x - 21 = (2x + 3)(x - 7)$. As a special case, a may equal 1, giving cases like $x^2 + 3x - 28 = (x + 7)(x - 4)$. For practical work the importance of this general type is much exaggerated, the type being needed only in cases of fractions or equations specially made up to illustrate its use.

13. *Ability to factor expressions like $a^2 - b^2$.* This case has value in simplifying certain formulas so as to change them to forms which are better adapted to computation.

14. *Understanding why division by 0 is not permitted.* It is important to understand this fact clearly, it being necessary when, in checking, a divisor becomes zero.

15. *Understanding the significance of complete factoring.* That is, if required to factor $x^4 - a^4$, it is not sufficient to write $(x^2 + a^2)(x^2 - a^2)$. We should write $(x^2 + a^2)(x + a)(x - a)$.

16. *Ability to apply the laws of exponents.* These laws are expressed as follows:

$$a^m a^n = a^{m+n},\quad \frac{a^m}{a^n} = a^{m-n},\quad (a^m)^n = a^{mn},\ \text{and}\ (ab)^n = a^n b^n.$$

In simplifying formulas they are of great value.

17. *Understanding of zero, negative, and fractional exponents.* The zero exponent is of no practical value in our work at present, but the negative and fractional exponents are often seen in formulas.

18. *Ability to check all results.* This is one of the essential things in algebra. Every time a formula is derived, the work should be checked. The most convenient check is the one in which some small numerical value is substituted for each of the letters.

19. *Understanding of the generality of algebra.* This means that the pupil should be led to see that, for example, the identity $(a + b)^2 = a^2 + 2ab + b^2$ includes an infinite number of special cases such as the following:

$$(3 + 2)^2 = 3^2 + 2 \cdot 3 \cdot 2 + 2^2,$$

or

$$5^2 = 9 + 12 + 4 = 25.$$

F. Fractions

1. *Understanding that a fraction means division.* This means that there are various ways of considering a fraction like $\frac{3}{4}$. We may think of it as three of the four equal parts of unity; as a fourth of 3; as the ratio of 3 to 4; or as an indicated division of 3 by 4. While all of these lead to the same result, it is simpler in algebra to think of the fraction $\frac{a}{b}$ as indicating the division of a by b.

2. *Understanding the principle of signs.* This means that the pupils should be so trained as to have no hesitancy in seeing that

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b}.$$

3. *Ability to reduce a fraction to lowest terms.* This means that the pupil should simplify all formulas as much as possible in this way. To take a familiar illustration, we know that the area of a circle may be expressed either as πr^2 or as $\frac{1}{2} rC$. Whence

$$\frac{1}{2} rC = \pi r^2,$$

or

$$C = \frac{\pi r^2}{\frac{1}{2} r}.$$

Now it is manifestly undesirable to leave this fraction in such an awkward form, and hence it should be reduced to lowest terms and simplified as much as possible. It is, however, of no practical value to devote much time to reducing such fractions as

$$\frac{x^2 + 6x + 9}{x^3 + 9x^2 + 27x + 27}.$$

They almost never enter into practical algebra.

4. *Ability to perform other reductions.* This means that there should be a moderate amount of work requiring such reductions as the following:

$$\frac{ax + b}{x} = a + \frac{b}{x}$$

and

$$3a - \frac{b}{4a} = \frac{12a^2 - b}{4a}.$$

The purpose of such work is to simplify formulas or to get them into a form better suited to use in deriving other formulas. The reduction to a L.C.D. is necessary in connection with simple

fractions having monomial or even binomial denominators. The reduction of or operating with fractions like

$$\frac{x^2 - 2x - 15}{x^3 - 125} \quad \text{and} \quad \frac{x^2 - 9}{x^3 - 5x^2 - 9x + 45},$$

however, is useless from the standpoint of practical algebra or of the theory of the subject. However unimportant such work may be with respect to reduction, addition, or subtraction, it is, if possible, of even less practical importance in multiplication and of still less in division. Such cases are merely made up to illustrate a principle; they so rarely arise in a practical problem as to have only a theoretical value.

5. *Ability to simplify such complex fractions as may be needed in work with practical formulas.* This requires, in elementary work, no more difficult cases than the following:

$$\frac{a + \frac{b}{c}}{a^2 + \frac{b^2}{c^2}} \quad \text{or} \quad \frac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}.$$

6. *Understanding of the connection between arithmetic and algebraic fractions.* It means a great deal to see how such relations as

$$\frac{a}{b} + \frac{x}{y} = \frac{ay + bx}{by}$$

express laws that we continually use in arithmetic. For example, the above is a formula that applies to all such additions and subtractions as $\frac{2}{3} + \frac{3}{8}$ or $\frac{5}{8} - \frac{3}{5}$, to take two cases of greater difficulty than those ordinarily seen in practice.

7. *Ability to check all results.* For example, in the above formula let $a = b = x = y = 1$, and we have

$$\tfrac{1}{1} + \tfrac{1}{1} = \tfrac{2}{1} = 2,$$

and so there is probably no error.

G. Fractional Equations

1. *Ability to clear an equation of fractions.* This means that in any equation in which either numerical or literal fractions enter the pupil should be able to write an equivalent equation with-

out any fractions. It is not, however, always desirable to do this, as was formerly the case when pupils were afraid of fractions. There is no sense in clearing of fractions an equation like

$$\tfrac{3}{4}x + 21 = \tfrac{5}{8}x + 29.$$

Any pupil who has progressed as far as the junior high school should be able to solve it without using pencil and paper. We need not be afraid of the expression "clear an equation of fractions"; after its meaning is plain it serves a good purpose.

2. *Ability to solve numerical equations containing fractional coefficients.* These may be either common or decimal fractions, as in the following cases:

$$\tfrac{3}{4}x = 7, \qquad 0.25\,x = 4 - x, \qquad x + 6\%\,x = 12.$$

3. *Ability to solve fractional equations.* This means, in general, equations with monomial or binomial denominators. Even in these simple forms, we do not often need the binomial denominator in more than one fraction in our work with elementary formulas. This work is, for our present purposes, usually limited to cases like this: (1) Solve for p the equation

$$A = p(1 + rt);$$

(2) Solve for s the equation

$$\frac{sr}{a} = r^n - \frac{a - s}{a},$$

each of which leads to formulas that are used in algebra.

4. *Ability to derive one formula from another.* We have frequently met with this case, and it is mentioned here again because the work very often requires the use of fractional equations of a simple type.

5. *Understanding the generality of a literal equation.* This applies to integral as well as to fractional equations. Any literal equation permits of an unlimited number of numerical equations of the same family and having the same form of solution. For example, if

$$ax + \frac{a}{b} = b,$$

then

$$x = \frac{b - \dfrac{a}{b}}{a} = \frac{b^2 - a}{ab}.$$

Without solving again, we may therefore substitute in the formula and see that the equation

$$3x + \tfrac{3}{4} = 4$$

has for its root $$x = \frac{16 - 3}{12} = \frac{13}{12}.$$

6. *Ability to solve for a constant an equation like* $y = ax + b$. This means that it is sometimes desirable to express the value of a or b in terms of the other three letters. For example,

$$a = \frac{y - b}{x}.$$

7. *Ability to evaluate formulas involving fractions.* This is the most important part of our present work in fractions, and the simplest.

8. *Ability to check all results.* This usually involves either the substitution of an integral expression or a fraction in an equation of the first degree, or the substitution of a number in place of letters.

H. Ratio, Proportion, and Variation

1. *Understanding a ratio as an abstract quotient.* This means that the quotient of any number divided by another of the same denomination is the ratio of the first to the second. For example,

$$\frac{3\text{ ft.}}{4\text{ ft.}} = \frac{3}{4}, \qquad \frac{4\text{ in.}}{2\text{ in.}} = 2, \qquad \frac{\$5.25}{\$0.25} = 21.$$

2. *Understanding a proportion as an equality of ratios.* This means that a proportion is merely a fractional equation and hence it should be treated as such. For example, if we have

$$\frac{x}{a} = \frac{b}{c},$$

we may solve the equation and find that

$$x = \frac{ab}{c}.$$

There is no reason for speaking of means, extremes, antecedents, or consequents, as was formerly the custom. Such words are merely useless lumber in our modern algebraic structure.

3. *Understanding that a ratio, written in fractional form, is subject to all the laws of fractions.* The old way of writing a ratio as $a:b$ is objectionable in our work, except as it must be known before a pupil reads books on physics. These books are generally very conservative, and they fail to use the much simpler fractional form now generally taught in algebra.

4. *Understanding variation as related to ratio.* This means that if x varies directly as y, we have the ratio

$$\frac{x}{y} = k,$$

some constant (k) therefore being the ratio of x to y. If, however, x varies inversely as y, then

$$\frac{x}{\frac{1}{y}} = k, \text{ some constant,}$$

or

$$xy = k,$$

or

$$x = \frac{k}{y},$$

which is also a ratio. Hence, in any case, a variation involves a ratio. Indeed, if it were not for the convenience of the term "variation," it might easily be abandoned. The relation of two variables should be considered graphically as well as symbolically; that is, we should consider with the pupils the graph of the equation $x = ky$ (where k is a known number) as well as the fact that k is the ratio of x to y.

5. *Ability to solve problems in variation.* This means at this time that the pupils should be given simple and genuine problems in both direct and inverse variation, even if the subject is not treated in the textbook in use.

6. *Understanding more fully the idea of function.* This means that if x varies directly as y, then

$$x = ky, \text{ where } k \text{ is a known number,}$$

and hence that x depends upon y for its value; that is, x is a function of y. It is not necessary, however, to dwell at length upon this feature in this connection.

I. Simultaneous Linear Equations

1. *Understanding types.* This means that, by the aid of graphs, the pupils should come to appreciate the significance of simultaneous, inconsistent, and equivalent equations in two unknowns. Such equations may be represented, respectively, as follows:

$$\begin{array}{ccc} x + y = 5 & x + y = 5 & x + y = 5 \\ 3x - y = 3 & 2x + 2y = 7 & 3x + 3y = 15 \end{array}$$

The graphs of these three will illustrate the types.

2. *Ability to choose the best method.* This means that there are three convenient methods of solution. These are (1) addition, (2) subtraction, and (3), most important of all, substitution. They are advantageously used in the following sets respectively:

$$\begin{array}{ccc} x + y = 5 & 3x + y = 9 & x + 3y = 11 \\ 3x - y = 3 & x + y = 5 & y = 3 \end{array}$$

3. *Ability to solve applied problems.* This means that the pupil should be able to use his knowledge of simultaneous equations in practical work. It will be found, however, that this field is rather limited unless we introduce fictitious problems or else presuppose more technical knowledge of science, industry, or commercial affairs than the pupil usually has. This is evident when we come to examine most of the current textbooks.

4. *Ability to check all results.* The only complete check is the substitution of the roots in both the original equations, showing that each reduces to an identity.

J. Powers and Roots

1. *Understanding necessary terms.* This means the understanding of the meaning of the terms *power, root, principal root, real number, rational number, irrational number, imaginary number, exponent,* and *index* of a root. There is also some value in the term *base.* It is not necessary to memorize the definitions of such terms. They will be understood through use.

2. *Seeing the practical need for roots.* It is of little value to learn about roots unless we are going to use them. It is, there-

fore, well to give the formula $A = \pi r^2$ to be solved for r. The result is $r = \sqrt{A \div \pi}$. We therefore see the desirability of knowing (1) that the reciprocal of π (that is, $1/\pi$) is 0.301, and (2) that we then need to find the square root of $0.301A$.

3. *Ability to find the square root of a number.* This means that a pupil should know (1) how to use a square-root table; (2) how to find a square root without a table, either by the formula for $(a+b)^2$ or by some such method as "trial and error." He should also know how to find an approximate root from the graph of $y^2 = x$, or $y = \sqrt{x}$.

4. *Ability to find the square root of a polynomial.* This means to find the square root of such polynomials as $x^2 + 2\,xy + y^2$ and $x^2 + 6\,x + 9$, the sole purpose being to aid the pupil in understanding arithmetic square root. The work may be extended to the case of

$$x^2 + 2\,xy + y^2 + 2\,xz + 2\,yz + z^2,$$

but there is no practical need of continuing farther.

5. *Understanding how far to carry a square root.* This means that in the case of a surd like $\sqrt{2}$ there is no end to the possible number of decimal places. When directed to carry to three decimal places, this means to the nearest thousandth. The value of $\sqrt{2}$ to three decimal places, or to four significant figures, is 1.414. The term "significant figures" means any one of the figures 1, 2, 3 . . . 9, and also 0 whenever it is known that the place it occupies is really a zero. For example, in the number 205 the 0 is significant; but when we say that the distance to the sun is 92,000,000 miles, we mean that we are not certain of more than two figures, so that the zeros are not significant in the ordinary use of the term.

6. *Understanding common exponential symbolism.* This means that the pupil should understand not only that $a^2 = aa$, $a^{-n} = 1/a^n$, and $a^0 = 1$, but that $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^m a^n = a^{m+n}$, $a^m/a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, and $(a^x b^y)^n = a^{nx} b^{ny}$.

7. *To apply square root.* This means that square root is to be applied in the formulas and in simple problems relating chiefly to the Pythagorean Theorem.

8. *To solve radical equations.* This means that the pupil should

be able to solve such radical equations as are needed in ordinary formulas. To prepare for this work, abstract cases may be given of no greater difficulty than $\sqrt{2x+1} = x - 1$.

K. Quadratic Equations

1. *Ability to construct and interpret the graph of a quadratic function.* This means to construct and interpret the graph of a function like $y = 2x^2 - 3x + 1$. It should be constructed, interpreted for the case of $y = 0$, and used to find the actual or approximate roots of the equation $2x^2 - 3x + 1 = 0$. The pupil should see that this is a special case of the general quadratic function $y = ax^2 + bx + c$, and that the latter represents an entire family of such functions.

2. *Understanding of the terms "complete" and "incomplete."* This means, these terms as applied to a quadratic equation. The older terms "pure" and "affected" have gone out of general use.

3. *Ability to solve a quadratic equation by factoring.* This means that if $x^2 - 2x - 63 = 0$, then $(x+7)(x-9) = 0$, whence either $x + 7$ or $x - 9$ may equal zero, and so $x = -7$ or 9. Any quadratic equation can be solved by factoring if we extend "factor" to include irrational numbers; but the operation usually becomes too difficult for elementary pupils. The only equations that they can solve in this way are simply made up for this purpose, not representing any genuine applications of algebra. It should be understood that the introduction of quadratic equations in the junior high school is purely optional.

4. *Ability to solve by completing the square.* There are two practical ways of quickly solving any quadratic. The first is the method of completing the square, this requiring the memorizing of the fact that, in the equation $x^2 + px = -q$, we must add to both sides (members) the square of half the coefficient of x. In the case of the quadratic $ax^2 + bx + c = 0$, we may, if we wish, reduce it to the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or

$$x^2 + \frac{b}{a}x = -\frac{c}{a},$$

and then we may complete the square as before.

The second method is to use the standard formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which we obtain if we solve the equation just given. This requires that we memorize the formula.

Each method therefore requires memorizing. For practical purposes there is little choice. The actual numerical work is the same in one case as in the other.

5. *Ability to find the maximum or minimum value of a quadratic function.* This may be done in two convenient ways within the reach of the pupil at this time. For example, in the equation

$$y = x^2 + 6x - 2,$$

we may draw the graph and find from it that the smallest value of y is -11.

The second plan is to observe that

$$x^2 + 6x - 2 - y = 0.$$

Solving for x, $\quad x = -3 \pm \frac{1}{2}\sqrt{36 + 8 + 4y}$

$$= -3 \pm \tfrac{1}{2}\sqrt{44 + 4y}.$$

Evidently the smallest that $4y$ can be is -44; for, if it were smaller, we should have a negative number under the radical sign, which would give us an imaginary number. Therefore the smallest that y can be is -11.

6. *Ability to solve applied problems.* There are only a few types of genuinely real problems that are simple enough for pupils in Grade IX and that require quadratic equations. Such as we have may be found in the better kind of modern textbooks.

7. *Ability to check the solutions.* In simple cases this is best done by substituting the values of the supposed roots in the equation. If the substitution is difficult, the best check is to observe that in the equation $x^2 + px + q = 0$ the sum of the roots is always $-p$ and the product is q. For example, the roots of the equation $x^2 - 7p + 12 = 0$ are 3 and 4, because

$$3 + 4 = -(-7) = -p, \quad \text{and} \quad 3 \times 4 = 12 = q.$$

Such a check is easily applied by any ninth-grade pupil.

9. Abilities in Numerical Trigonometry

General Purpose. Trigonometry can be made very difficult or very easy, and so can every other subject. For the junior high school it should not be as difficult as the relatively useless part of algebra that it replaces. The general purpose in teaching it is to show the pupils the significance of indirect measure — of measuring the distance across a river, for example, without going from one side to the other. If presented simply, as a practical subject, it is both easy and interesting besides having a value that is unquestionable. It is the basis of all measurements of land on the earth's surface, of the size of the earth itself, of the distances to other heavenly bodies, and of a good part of our great pieces of engineering.

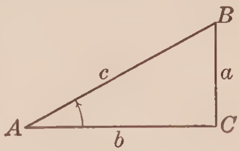
It should be the purpose of the school to show the significance of all this in the life of today, using only those functions and formulas that are necessary for this purpose.

Classification. The subject is taught for so short a time in the junior high school that the classification must necessarily be limited in extent. The following topics should be considered:

A. Necessary Functions

1. *Four important functions.* In trigonometry there are several possible functions of an angle. Of these the beginner needs only four, — the sine, cosine, tangent, and cotangent. We could easily reduce the number even more, but these four are desirable in the introductory course.

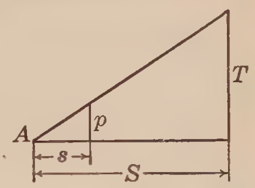
2. *Definition of functions.* It is possible to define these functions as lines, but it is usually thought more convenient to define them at first as the following ratios:



$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a}.$$

3. *Introduction to the functions.* It is advisable to introduce the functions by beginning with the tangent. For example, to take a case that always interests pupils, if a tree T feet high

casts a shadow S feet long at the same time that a post p feet high casts a shadow s feet long, we have the proportion



$$\frac{T}{S}=\frac{p}{s},$$

whence $$T=\frac{p}{s}\cdot S.$$

That is, we can find the height of the tree by multiplying the length of its shadow (S) by the ratio $\frac{p}{s}$. Thus we can find the height of the tree indirectly; that is, not by climbing the tree and measuring with a tape, but by using mathematics.

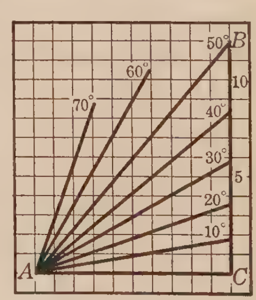
The ratio $\frac{p}{s}$ is called the *tangent* of $\angle A$; that is,

$$\tan A=\frac{p}{s}.$$

We then can find the length of a in the first of these figures by observing that

$$a=b\tan A.$$

4. *Ability to find the functions.* It is possible to find the tangents of angles of various sizes by means of squared paper, as here shown. In practical work, however, we use tables which have been computed for us by methods of higher mathematics and which give the tangents of the angles correct to three or more decimal places.



On a piece of squared paper we may lay off angles of 10°, 20°, 30°, · · · at A, as here shown. Then, taking $AC=10$, we may draw CB. The tangent of each angle is then the number of spaces cut off on CB, divided by 10. We can thus find approximate values of the tangents as follows:

$$\tan 10°=\frac{1.8}{10}=0.18 \qquad \tan 30°=\frac{5.8}{10}=0.58$$

$$\tan 20°=\frac{3.6}{10}=0.36 \qquad \tan 40°=\frac{8.4}{10}=0.84$$

The pupil should do this work for the tangent, the sine, and the cosine, and thereafter should use the tables in finding any required function.

5. *Ability to use any convenient tables.* For the beginner, either three-place or four-place tables may be used, according to whichever the textbook offers.

B. Applications

1. *Ability to apply the subject to solving right triangles.* There are more interesting and valuable applications of simple trigonometry than of any other part of elementary algebra, perhaps excepting formulas. The work should be applied to actual outdoor measurements, such as finding the height of a tree, the school building, or a church spire.

2. *Instruments needed.* If the school has a transit, this should be used. If it has not, the pupils can easily make, as they do in very many schools, simple instruments for measuring angles, using a pointer turning on a paper protractor.

3. *Ability to check.* All work in mathematics should be checked. A desirable and convenient check in trigonometric measurements is achieved by comparing and discussing the different results in class.

QUESTIONS AND TOPICS FOR DISCUSSION

1. What is the peculiar function of the junior high school? How does this affect the objectives in mathematics?

2. Indicate any omissions or additions which you would make in stating the following, the numbers in parentheses referring to the sections in this chapter:

   a. Great central mathematical objectives (2).
   b. General objectives (2).
   c. Psychological objectives (3).
   d. Objectives related to the seven cardinal objectives (4).
   e. Concepts of elementary mathematics (5).
   f. Abilities in arithmetic (6).
   g. Abilities in intuitive geometry (7).
   h. Abilities in algebra (8).
   i. Abilities in numerical trigonometry (9).

3. Give the arguments for and against project teaching in the junior high school.

4. Review the list of criteria which you consider valid to use in choosing objectives in mathematics for the junior high school.

5. If you were required to take care of wide differences in ability among pupils, how should you proceed to discriminate between a list of minimum-essential objectives and a more elaborate one?

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CHAPTER IV

THE TEACHING OF ARITHMETIC

1. General Purposes in View

Purpose of the Chapter. Before considering in more detail the methods of attaining the objectives listed in Chapter III it is desirable to take a more general view of the situation and to ask ourselves once more what is the great central purpose in mind when we require pupils to spend the greater part of eight years in studying arithmetic. The first part of this chapter is, therefore, not so much a study of particular objectives as it is of general principles. While the reader will naturally expect that special objectives and general principles will frequently overlap, as must necessarily be the case, it will be found that this repetition merely emphasizes certain important points, and that each method of presentation supplements and strengthens the other. In particular, we shall lay more stress at first upon the elimination of nonessentials. Furthermore, we shall take a broader view as to the general nature of arithmetic than we did in Chapter III, where we dwelt more upon the bare essentials to be retained and the fundamental skills to be developed. This being accomplished, we shall discuss the objectives in such detail as will permit the teacher to make it reasonably certain that the necessary ground is being fully covered. If we are to develop certain arithmetic abilities in our pupils, so that they shall be able to use them accurately in their daily lives, we must keep our objectives clearly in mind. In addition to this we must develop these abilities until our tests show that the pupils have mastered them.

No attempt has been or will be made to introduce the technical information necessary to understand the more complex arithmetic of the worker in the machine shop, of the manu-

facturer, or of the scientist. Such knowledge is important, but it is special. We shall include only those items of information that are necessary in the pupil's general equipment.

The Scope of Arithmetic. We do not think it strange to see the subject of taxes included in our courses of study. Since it is there, we expect to find it in our textbooks, which naturally offer what the standard courses demand. For two or three centuries our arithmetics have included the topic, and fashion has decreed that they should continue to do so. No course of study, however, has anything to say about trade-union dues, how they are assessed or how they are spent — a perfectly legitimate feature of modern business life. These two topics serve to illustrate the fact that the scope of arithmetic is a matter of custom quite as much as of human needs. The strictly arithmetical part of what we call arithmetic is relatively small; the real or fancied applications have come to occupy the major part of the pupil's time, at least in Grades VI–VIII.

The result of this extension of the work in arithmetic to include a certain amount of political economy, of business practice, of sociology, of general science, of investments, and the like has been to obscure the real objective already stated — to train the child to compute accurately, with reasonable speed, in those operations that he will probably use in later life.

Now this extension of the scope of the subject so as to include various types of application is proper provided we select those topics which are of most value to the prospective citizen. Evidently, however, it is impossible to include the whole range of human interests in which computation plays a part. A child of twelve or thirteen in a village in Louisiana is not likely to develop a very enthusiastic interest in the computing of the tax rate in Chicago, nor is one who lives in the poorer parts of New York City likely to be attracted to arithmetic by problems on the devastation caused by the boll weevil. It is equally doubtful whether, in Grades VII and VIII, children will get much more out of the arithmetic relating to calories and vitamins than they will get out of problems about electrons in relation to the radio.

It therefore becomes necessary, in the work of improving our arithmetic, to weigh with much care the topics that are required in our standard state and city courses, as well as those which are beginning to claim a place on the ground that they represent modern conditions.

2. The Nonessentials in Arithmetic

The Nonessentials in General. In every branch of human knowledge there is certain material that is essential to the mental equipment of every fairly-well educated citizen. Every part of human knowledge is essential to someone, but some parts are essential to everyone. It is necessary that we should all know the product of 2 and 3, but it is necessary that only a very few should know the cube root of 2 to three decimal places. It is convenient to certain persons to remember throughout their active lives that $\frac{7}{8} = 0.875 = 87\frac{1}{2}\%$, but this is not an essential part of the equipment of the great mass of people, although it is required by many courses of study and may properly be taught as a matter of numerical gymnastics.

Similarly, in the training of a teacher certain parts of the work ordinarily given are essential, but a large portion of the work which is given in our schools of education has only a temporary value: it is not of fundamental importance to all teachers, and in time much of it finds its place in the great necropolis of forgotten theories.

We are prone to accept as essential those parts of knowledge which have held place in the schools for a long time, without considering the rapid changes in business and in social demands. On this account some of the parts of arithmetic that we shall class as nonessentials will strike many teachers as fundamental in the education of children, — as, indeed, they were at one time. It is, however, only by facing the issue with perfect candor that we shall be able to decide what parts of the subject must be required for all in the next quarter of a century; and it is only by calling attention to the parts of arithmetic which are approaching the status of the nonessential that we can hope to keep them out of the tests, the educational psychology texts,

the state and city syllabuses, and the examination systems, all of which combine in some degree to form the last stronghold of ultraconservatism.

In listing the nonessentials of arithmetic there is no need for mentioning such topics as equation of payments, cube root, series, troy weight, and the like; to do so would be simply to set up a man of straw for the pleasure of attacking him. They are no longer found in our modern texts, except as chance bits of information that is no longer required and that is practically dead so far as the essential features are concerned. The inquiry must go beyond this if it is to have any value.

Basis of Judging the Essential. As we have already pointed out, the basis of this inquiry cannot be the official courses of study, these being in general more or less reactionary. It cannot be the current textbooks, even the most progressive ones; these are bound by the necessity of covering the topics which the courses of study lay down, although the best of them continue to urge the neglect of the nonessentials which they are forced to include.

The basis of the inquiry must be a knowledge of present social demands. What parts of arithmetic have the people as a whole ceased to use? The special needs of the special worker, say in the shop, on the farm, in medicine, or in the bank, though they must form a part of the technical education of these specialists, are nonessentials in a general course in arithmetic for everybody. The great question to be considered relates to the essentials in the equipment of the average citizen and to the nonessentials as they appear in our courses of study at the present time.

Among the latter that still remain in most of our courses of study in arithmetic and in certain of the treatises on the psychology of the subject are a few that rather clearly must disappear in the near future, and these may first demand our attention. In considering them it must be recognized that the child may profitably be carried beyond the limit of the absolute demands of elementary business, but it should also be recognized that this argument has been used in the past to sanction the re-

tention of every obsolete topic in every subject in the schools, and that it will be used as an argument against every suggestion for the omission of the nonessential from our courses of study, our tests, and our psychological treatises. We shall, therefore, consider first the nonessentials which, it would seem, must soon disappear from these sources of authority, thus releasing the textbook writers from the unfortunate necessity under which they labor. Let it not be thought, however, that our purpose is destructive criticism — the favorite subject of the educational demagogue. The purpose is distinctly to show that by judicious limitations there will be plenty of time for drill upon the basic number skills and at the same time for such topics as intuitive geometry, numerical trigonometry, and the newer type of elementary algebra.

Nonessential Definitions. In the matter of definitions there has been a great gain in the last generation. The child is no longer required to memorize the definitions of such terms, for example, as "notation," "numeration," and "addition." He simply learns to use the terms correctly, just as he uses the words "fork," "house," and "fire," no one of which he is expected to define with any degree of accuracy. In fact, the whole subject of memorized definitions in arithmetic has already been placed in the category of the nonessential. In spite of all this, however, the schools still make too much of the definitions of such terms as "fraction," "numerator," "percentage," and "interest." There is a good reason for memorizing the definitions of a few terms used in geometric proofs, but so far as arithmetic is concerned it would be a good thing if no child were ever required to memorize any definition whatsoever. A more complete discussion of definitions will be found on pages 247–252.

Nonessential Terms. As to the terms themselves, there has also been a great advance in recent years. Such terms as "notation," "numeration," "composite number," and "compound number" have substantially disappeared. There remains much more work to be done, however. As will be asked again, when we come to consider the detailed objectives, why should children in the primary grades be asked to remember the distinc-

tions between "sum," "difference," "product," "quotient," and "remainder," when the single word "result" answers all their simple purposes? Such changes cannot be made at once; words like "sum" and "quotient" are well entrenched, and so long as people use them generally, the schools should retain them. But how many teachers are certain as to when and where to use the word "remainder"? The word "product" was quite commonly used for "sum" not so very long ago, and in practical life the word "difference" is very rarely used as it is in a class in arithmetic. Indeed, in practical life we use "deduct," "take off," "take away," or simply "take," instead of "subtract," and we still retain the older word "rest" instead of "difference," as when we say "take out what I owe and pay me the rest." Anyone would be laughed at, in business, if he should say "subtract what I owe and pay me the difference." The word "dividend" is ordinarily used in business in a sense quite foreign to the one taught in school. All this suggests that we shall probably come more and more to speak of "the numbers added" instead of "the addends" and of "the number from which we subtract" rather than "the minuend," while retaining such simple words as "multiplier" and "divisor," these being in general use in daily life. No wholesale discarding of such technical terms is advisable or possible, but terms like "subtrahend," "dividend," and "multiplicand" are not very important in the schools, and there seems no doubt that they will cease to perplex the learner after this fact is fully realized.

Nonessential Concepts. Obsolete concepts are also disappearing with satisfactory rapidity, but we still have numerous expressions that, in the modern teaching of elementary arithmetic, are sure to be discarded in the near future. These include such terms as "prime number," "composite number," "greatest common divisor," "least common multiple," "least common denominator," "involution," "evolution," "means and extremes" of a proportion, "antecedent and consequent" of a ratio, and possibly "digit." Each of these has served its purpose and would have been abandoned some time ago by progressive textbook writers had they been free to follow their inclinations.

Nonessential Computations with Whole Numbers. Our social life of today demands that every citizen with a fair education should know how to write and read numbers to billions; to add columns of a few numbers, say eight or ten, these numbers being confined chiefly to dollars and cents, the need rarely extending beyond $100 in the case of the average citizen who is called upon to check his bills; to multiply numbers within the same limits by numbers of one or two figures; and to divide by numbers of one or two figures. Anyone who can perform these operations will be able to perform the same operations with larger numbers if the necessity arises after leaving school. His added maturity will take care of the difficulty involved. If we search for absolute essentials with whole numbers or with numbers involving dollars and cents, we cover them in the above list.

That good teaching requires the passing somewhat beyond these limits on special occasions will hardly be questioned by anyone; but if we consider the absolute essentials to be required by a course of study or to be covered in general tests, they are included in the above statement and will be considered later in further detail.

We may therefore list a few nonessentials that may safely be omitted except for occasional extensions in teaching, the purpose of including them in actual classroom practice being to arouse interest or to carry a pupil beyond the limits of everyday business. Looking to the ordinary needs of the ordinary person, the list would then include such work as the following: reading and writing numbers beyond billions; the addition of abstract numbers of more than three or four figures and in columns of more than eight or ten; the addition of dollars and cents beyond thousands of dollars, the ability to add larger numbers developing naturally with the added maturity of the pupil as the need arises; the addition of such abstract numbers as those here shown, work of this kind, with abstract numbers, entering so rarely into the experience of any one as to be quite nonessential; subtraction of such types as those suggested in the above con-

$$\begin{array}{r} 4273 \\ 4 \\ 207 \\ 36 \\ 1204 \\ 3 \\ \hline \end{array}$$

sideration of addition; multiplication by three-figure numbers; multiplication of long abstract numbers, such as $37 \times 92{,}836$, but not including such cases as $37 \times \$928.36$, even though such an extensive multiplication is a rarity with most people; division of long abstract numbers; and division by any integer of more than two figures.

It should be repeated and clearly understood, however, that for purposes of teaching it is entirely proper to extend the work reasonably beyond the limits here set. What is attempted above is to set forth a partial list of nonessentials considered with respect to ordinary needs, to the making of a syllabus, and to the judging of a set of tests. If we should concentrate more attention on the essential computations and less on the nonessential, we would make better arithmeticians in the domain of ordinary daily business.

Nonessential Computations with Decimals. Using the word "decimal" in the modern sense, to mean a decimal fraction, and excluding for the present such decimals as represent dollars and cents, let us consider what work in this subject may be classed among the nonessentials.

In the first place it is quite evident that we never operate upon abstract decimals, as in the cases of 3×4.278 or 2.483×0.9827. Except as arithmetical gymnastics such work has little value to the average person. We use decimals only in connection with measures of some kind. We may pick up a book 37 times, but we cannot pick it up 3.7 times. We may wish to find $\frac{1}{16}$ of a number, but we never wish to find $\dfrac{1}{1.6}$ of it. Our essential operations with decimals are much more limited than those with whole numbers.

In writing and reading decimals the needs of the average citizen are covered by three decimal places; indeed, two places answer his purposes fairly well, and even here his chief uses are confined to dollars and cents.

For the addition of decimals the average citizen has no need, always excepting dollars and cents as above stated. In special cases decimals are needed in connection with meas-

urements, chiefly of length and weight, but the operation is so similar to that involving money as to offer no difficulty whenever the individual has need for it. That the subject is to be eliminated from the schools is not, of course, suggested; what is suggested is that it cannot rank as an essential in the same sense as the addition of \$1.75 and \$2.50, and that a considerable part of the work set forth in city syllabuses and in many current tests is useless. This may be illustrated by the absurd case here given, which is of the type so often seen at the present time. It would be impossible to find anything more unreal in the whole domain of arithmetic. Since decimals are used, as already stated, only in recording measurements of some kind, it is evident that if we measure to ten thousandths in one case in a problem, we should do so in all cases. If such an addition ever occurred at all, it would be given in the form here shown; it would never be given as set forth above, where the effect is merely to puzzle the pupil, although this was not the intention.

$$\begin{array}{r@{}l} 2&.7 \\ 12&.836 \\ &.94 \\ 127&.1 \\ 3&.0076 \\ \hline \end{array}$$

$$\begin{array}{r} 2.7000 \\ 12.8360 \\ 0.9400 \\ 127.1000 \\ 3.0076 \\ \hline \end{array}$$

What has been said with respect to addition applies with equal force to subtraction. Not only is an example like $12.7 - 4.0396$ a nonessential, but it is an evidence of educational ignorance. Such work is still demanded in enough courses of study to compel textbook writers to include it to some extent, but it should soon disappear from our schools. Even when we consider the case of $1\frac{1}{2} + 2\frac{7}{8}$ it is understood that each measure is accurate only to the nearest eighth, and if the example is expressed decimally it properly appears as $1.5 + 2.9$ to the nearest 0.1, $1.50 + 2.88$ to the nearest 0.01, or $1.500 + 2.875$ to the nearest 0.001; and similarly in the case of subtraction.

With respect to the multiplication of decimals, few citizens ever meet with a case like 2.37×96.483. They meet with $2\frac{7}{8} \times \$2.40$ but not with $2.875 \times \$2.40$, although the two give the same result. In technical work in mechanics, engineering, and the like a man will need such cases, and he will learn them then; but for most people they are simply nonessential. If the

diameter of a wheel is 14.7286 inches (an exceedingly rare degree of accuracy, even in technical work), the circumference is 3.14159×14.7286, but the result will be accurate only to four (not nine) decimal places; for the average man and woman, however, such work is manifestly not essential and it should not be required by any general course of study. Indeed, for such a person (not for the technician) it may be said that multiplication by a decimal of any kind can hardly be a very essential feature. In finding the circumference of a cylinder, the multiplier $3\frac{1}{7}$ is so commonly used as to render multiplication by 3.14 unnecessary in most cases that arise.

In the case of division, where pupils are often puzzled in placing the decimal point in the result, it is quite safe to say that no division by a decimal is an essential in the training of most of our people except as it refers to problems in science. A certain amount of this work should be included in a course of study because of these demands of science and also as a matter of interest and of general information, but textbook writers should be permitted to reduce it materially, which at present the tests and the courses of study will not allow. Such a case as $1.43 \div 0.7862$, as a problem in the junior high school, is a gross absurdity; it never occurs in ordinary life and is exceedingly rare in this form in any of the sciences.

It will therefore be seen that much of the work in decimals as required in our recent courses of study is nonessential as an equipment for the work of the large majority of well-educated citizens. Teachers feel compelled to teach much of it because of the withering effects already mentioned of many courses of study and current tests, but if freed from these influences the schools might safely make a radical reduction in this work and devote the time to the essentials.

Nonessential Work with Fractions. Traditions die hard. There are many such inheritances in connection with fractions, — using this term in the modern sense to mean common fractions. It is gratifying to record the progress made in the last quarter of a century in the elimination of such forms as $\frac{125}{875}$ and such operations as $3\frac{3}{17} + 5\frac{9}{26}$. We have come to recognize the

fact that fractions are commonly used only in finding the fractional part of something or in making measurements. We need to find the cost of $2\frac{3}{8}$ yd. of velvet at \$4.25 a yard, but it is a rare thing for the average citizen to need to find the area of a piece of glass $2\frac{3}{8}$ in. by $4\frac{1}{4}$ in. We may even wish to find the cost per yard of $2\frac{3}{8}$ yd. of velvet for which we pay \$10.09, although this would be a very rare case, the cost per yard being known when we buy the goods. But to find the length of a piece of glass that is $2\frac{3}{8}$ in. wide and that has an area of $10\frac{3}{32}$ sq. in. would be a case of such rarity as to be practically nonexistent.

We therefore see that certain work in fractions, still required by our courses of study, is manifestly nonessential so far as most people are concerned. Let us, then, consider certain features that are still required by current tests and courses, and still subject to investigation on the score of difficulty by many students of psychology.

In the addition of fractions most people no longer have any practical need for denominators requiring any reference to a least common denominator. These cases are relics of the days before the invention of decimals. It is quite impossible to conceive of a practical case involving, say $2\frac{3}{11} + 1\frac{6}{7}$, and fortunately such work is now disappearing from our schools. If such a case should arise, any person of sense would express each as a decimal, would add as with whole numbers, and would give the result to the nearest 0.1 only. We have, however, in nearly all of our detailed courses of study, such work as $2\frac{2}{3} + 3\frac{1}{2}$ and $2\frac{2}{3} + 3\frac{3}{5}$, and we often fail to observe that this is substantially as unreal as the ones above mentioned. Our measuring rules are graduated by halves, fourths, and eighths, and sometimes by sixteenths; occasionally they are graduated by tenths; but they are not graduated by both halves and thirds. We weigh by pounds and halves or quarters, or even eighths and sixteenths (ounces), but we do not weigh by thirds. We can make up any number of examples illustrating $2\frac{2}{3} + 3\frac{1}{2}$, but it is probable that they will all be fictitious.

This being the case, we may safely say that examples like $2\frac{1}{2}$ in.$+ 3\frac{7}{8}$ in., or 3.1 in. $+$ 4.8 in. are essential; while those like

$2\frac{2}{5}$ in. + $3\frac{5}{8}$ in. are not so. In other words, if we should confine our addition of fractions to cases in which the denominators are 2, 4, and 8, with the possible inclusion of 3, we should meet the ordinary needs of life. Other cases in the addition of fractions might safely be relegated to the nonessential group or to some technical field, which will readily be opened when the individual enters some vocation which requires such types.

What has been said with respect to addition applies equally to subtraction: a case like $6\frac{1}{4}$ in. − $2\frac{7}{8}$ in. is real, but one like $6\frac{3}{5}$ in. − $2\frac{2}{3}$ in. is unreal and nonessential.

With respect to multiplication the case is somewhat different. We do not have to add $3\frac{1}{7}$ ft. to $24\frac{1}{2}$ ft., but it is entirely conceivable that we may need to take $3\frac{1}{7}$ times $24\frac{1}{2}$ ft., as in finding the circumference of a cylinder needed in any one of several practical cases. This, however, does not mean that $3\frac{1}{7}$ times $24\frac{14}{15}$ ft. is equally practical, for such a case is absurd. We may therefore summarize the matter by saying that the practical part of multiplication is covered by such cases as $\frac{2}{3}$ of \$4.35, $3\frac{3}{8} \times$ \$5.60, $\frac{3}{4}$ of $4\frac{1}{2}$ in., and (though far less essential) $2\frac{3}{4} \times 5\frac{7}{8}$. For most people the first three are ample, and in none of the ordinary walks of life does the ordinary business man have any need for cases like $3\frac{4}{11} \times 5\frac{3}{7}$. If we hold to fractions with denominators 2, 4, and 8, we cover most cases in measurements, and if we admit $3\frac{1}{7}$, or $\frac{22}{7}$, we cover the somewhat rare demand in circular measure. There remains, then, only the further case of dividing into parts, of which $\frac{2}{3}$ of \$4.35 is a fair illustration, and which in such rather rare cases as $\frac{7}{25}$ of \$1250 is easily covered by taking $7 \times \frac{1}{25}$ of \$1250. The detailed objectives in this work in fractions are considered on pages 108–110.

With respect to the division of a fraction it may be said that, in general, the subject is of little value to most people, and that it will in the course of time be rated as a nonessential. We occasionally need such cases in multiplication as $\frac{2}{3} \times \frac{3}{4}$ and $2\frac{1}{2} \times 3\frac{7}{8}$, but it is very rare in daily life to find cases like $\frac{2}{3} \div \frac{3}{4}$ and $2\frac{1}{2} \div 3\frac{7}{8}$. It seems lamentable that children should be puzzled in attempting to comprehend the reason why $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$ when the operation itself is rarely if ever used in practical life.

In a general way, therefore, we may rate as nonessential a considerable part of the work in decimals and fractions as required in our courses of study of today, and a certain part of the work with whole numbers. As already stated, it is entirely proper to teach some of this as a matter of interest and for the purpose of carrying the pupil beyond the limits of the merely essential, but as a required essential it has no place in a course of study or in tests set for children.

Nonessentials in Measures. There has been in the last generation a marked diminution in the number of measures in common use all over the world. In America we no longer teach apothecary's weights in the elementary school, leaving the subject for the special training of physicians and pharmacists, and recognizing that even here the metric system is making rapid progress. In the same way there have disappeared from the schools as nonessential for the great mass of people such measures as the barleycorn, fathom, long hundred-weight, long ton, furlong, league, line, chain, link, perch, and grain. There still remain, however, various measures that have little meaning for city people, such as the rod, square rod, and cord. Just how long these will continue to be taught in city schools at least, it is impossible to say, but they can hardly be called essential to the education of the urban child, even for purposes of general information.

On the other hand, the metric system was not looked upon as essential a generation ago, but at the present time every boy knows the meaning of the 100-meter and the 10-kilometer race, and he at least hears of the 450-meter wave-length in connection with the radio. Even in elementary work in general science he meets with the need for the millimeter, the cubic centimeter, the gram, the liter, and the kilogram. All these terms and their meaning can be learned in a few minutes. Indeed, if we eliminate the nonessentials, the metric system is exceedingly simple. The nonessentials include such units as the dekameter, hektometer, myriameter, dekagram, hektogram, myriagram, dekaliter, hektoliter, kiloliter, stere, are, dekare, and hektare. Such terms simply make the work seem difficult, and they are not

needed as part of a working knowledge of a system which is used in all scientific work the world over.

Nonessentials in Denominate Numbers. With the exception of the addition of feet and inches, as in the case of 2 ft. 4 in. and 3 ft. 9 in., we may quite safely say that few of the other operations with denominate numbers are essential to the general training of the citizen. In various special cases they are needed, and they are learned with the vocations which make use of them. Cases like $2\frac{1}{2}$ times 7 yd. 19 in., 8 bu. 2 pk. $\div$ $3\frac{1}{2}$, and 4 lb. 7 oz. + 9 lb. 2 oz. + 6 lb. 13 oz. represent conditions which, so far as most people go, are obsolete. Even the case of finding the difference in time, as from June 27, 1934, to February 4, 1941, has come to have but little value since the modern methods of lending money have come into use. To be on the safe side, however, we may say that the following cases represent about the limit in the operations of probable value to most people with respect to this type of number:

2 ft. 4 in. + 3 ft. 9 in.	2×3 yd. 16 in.
3 yd. 19 in. − 1 yd. 22 in.	6 yd. 9 in. $\div$ 2

From these cases are excluded such measures as 8 gal. 3 pt., 2 bu. 2 qt., 17 rd. 4 ft., and 9 lb. 3 oz., since for most people these have no practical value.

As to the numbers themselves, all cases of more than two denominations, that is, all such as 3 bu. 2 pk. 1 qt., may safely be excluded. The only exception that is likely to occur to anyone is that of finding the difference between dates for purposes of computing interest, as in the case already mentioned, but this is more rarely used (without tables) at the present time than apothecary's measures or troy weight, and for most people is of no special value.

It will therefore be seen that the essentials in the work in denominate numbers may be reduced to such proportions as to allow much more time for drill in the vital part of arithmetic — the computations of daily life.

Nonessentials in the Applications. The applications of computation reach every branch of human industry and economics.

Men who could not read or write have been fairly successful in business, but without some ability to compute this is impossible. In considering, therefore, the purposes or objectives of arithmetic it is evident that much care must be taken in selecting the practical applications lest the offering should become too bulky for the schools to handle or for the child's mind to grasp. In order to arrive at a basis for selection it is well to consider briefly certain changes in social demands and their effect upon the content of the curriculum.

Changes in Needs as Civilization Develops. At one time the two chief applications of arithmetic were taxation and barter (the trading of goods). With the invention of money, profit and loss came to occupy an important place, as did also exchange, partnership, and interest. All these topics were important thousands of years ago.

As civilization develops, however, new needs appear, industry becomes specialized and localized, and we find in arithmetics such topics as the Rule of Three (substantially proportion applied to commerce, a subject now obsolete in this form), discount, banking, stocks, equation of payments, partial payments, insurance, mixtures, problems of motion, investments, marine insurance, and the metric system. Arithmetic tends to cease being merely the art of computation which children can understand, and to become a treatise on economics, sociology, mechanics, banking, and general business, much of which they only vaguely comprehend. The applications change from generation to generation, equation of payments dropping out and graphs coming in, and similarly with other topics, but in any case the applied work is at present chiefly economical and sociological.

3. Specific Purposes

Dual Purpose in Teaching Arithmetic. The great central purpose in teaching arithmetic has more chance of being realized if two specific purposes are kept in mind by both teachers and pupils. They are as follows:

1. *Numerical.* To develop in the pupil the ability to compute accurately and with reasonable speed in the fundamental opera-

tions with whole numbers, decimals (chiefly dollars and cents), and a few common fractions. For example, he must know how to add 8 and 7, $\frac{1}{2}$ and $\frac{3}{4}$, \$2.55 and \$1.25.

2. *Social and civic.* Since many applications of computations have to be made by everyone, we should find out what important social needs are met by the study of arithmetic. We should then see that the pupil gets the training that will enable him to do efficiently what he will be compelled to do in some fashion or other anyway. This does not mean that we are to have all social content and no arithmetic, but that the two shall be properly blended. For example, the pupil must know the arithmetic connected with his understanding of a tax rate, thus combining computation and social needs.

Numerical Objectives. We consider the numerical or pure arithmetic objectives first because they are most essential and more clearly agreed upon. They may be classified for the purposes of this discussion into four groups as given below:

1. Whole numbers
2. Common fractions
3. Decimals and per cents
4. Denominate numbers

The classification is intended to serve the needs of pupils generally. It gives no thought to classifying the items to meet the needs of any local situation. Adjustments can be made to meet these needs even though the scheme given here is made the basis of the course.

Objectives with Whole Numbers. In the field of whole numbers we need to secure the following objectives:

1. *Counting.* In the grades before the seventh the pupil first learned to count objects of various kinds and afterwards to count without any reference to particular objects. Beyond this stage there is little need for the ability to count. The need for such work is limited to small numbers.

2. *Reading.* As far as the needs of most people are concerned, the reading of numbers may well be limited to billions. We might have occasion to read such a statement as the following: "In 70 years the heart beats 2,800,000,000 times," but very rarely that "The sun travels 9,300,000,000,000 miles in 25,000 years."

3. *Writing.* The writing of numbers may also be limited as in the case of reading already mentioned. If large numbers are written they are expressed in words by stating the number of zeros or by expressing them as multiples of a million, a billion, a trillion, or in general as multiples of some power of ten. It is hardly worth while to require a knowledge of number names beyond trillions. It is desirable that the pupil should know what the scientist now does in the case of larger numbers like 72,400,000,000,000; that is, that he writes 7.24×10^{13} for the number; but it is not essential that elementary classes should be drilled upon such informational matters.

4. *Degree of accuracy in writing results.* The pupils should be taught what it means to say "correct to hundredths," "correct to three decimal places," and possibly "correct to four significant figures."

5. *Addition.* The pupil should be able to give automatically the results for any of the ordinary addition combinations. This power of recognizing such results is secured by "mechanical efficiency through habit formation" and may be considered a mechanical objective. How such habits are best formed is a matter which relates to primary arithmetic and need not be considered here.

Another valid objective is to develop the ability to add quickly a column of from two to ten items where the numbers are like \$354.38. This, though it carries us into decimals, may properly be considered here.

6. *Subtraction.* The objectives in subtraction are so related to those in addition and are so simple as to need no further comment. The most advanced type of practical problem in subtraction may be illustrated by the following cases:

9	34	\$1562.83
2	18	561.09

7. *Multiplication.* In multiplication as in addition we have two related objectives. The first is to recognize that the results of 4×6 and 6×4 are the same, and so with all such combinations as are involved in the actual cases of multiplication that

the child may meet. The second is to realize that one may need to multiply $43.46 by 64, but it is probable that he will never need to multiply 5,684,725 by 1572. The latter is not a real case.

8. *Division.* The mechanical objective in division is to get accurate results in the simple cases like $36 \div 4$ and $82 \div 9$. The latter gives a remainder, but such cases must be considered because we need the ability in dividing a number like 7598 by 6 or in writing a fraction in the result. The upper limit of probable need is represented by the division of $7952.26 by 86, giving the result to the nearest cent. If the pupil should ever meet with any more difficult case, his maturity will lead him to seek the right way out. The most difficult cases needed today in statistical work are done by calculating machines.

9. *Use of short methods.* The pupil should be taught to use any of the legitimate "short cuts" that can be used to save time in the fundamental operations with whole numbers and with common and decimal fractions. In particular we need to develop the ability to use them in multiplying by 10, 25, 50, 100, 1000, $12\frac{1}{2}$, and $33\frac{1}{3}$, and less frequently by $16\frac{2}{3}$, $66\frac{2}{3}$, and 75. We also need the ability to use them in dividing by 10, 25, 50, 100, and 1000.

Such methods should not be stressed too much. If they can be justified to the pupil and can be repeatedly used to save time and energy, they should by all means be employed.

10. *Powers and roots.* There is practically no need for the great majority of people today to know how to compute any power or root. Such work is all done for us in tables. In algebra, where we seek to show the dependence of one number upon another, there may be some reason for graphing the relation between a set of numbers and their squares or their cubes, but there is no such need in arithmetic. Of course for specialists, working in their special fields, such operations are needed.

Objectives with Common Fractions. The common fractions which we still need to use are halves, thirds, fourths, eighths, tenths, twelfths, sixteenths, and in rare cases fifths, sixths,

sevenths, and thirty-seconds. In connection with such fractions the following items should be considered:

1. *Addition.* We need to develop the ability to add halves, fourths, and eighths, and occasionally thirds as in the case of having to add $\frac{2}{3}$ yd. to $\frac{7}{8}$ yd.

There may be a probability of having to add in such cases as $2\frac{3}{4}$ lb. + $5\frac{1}{2}$ lb., or $15\frac{1}{2}$ in. + $7\frac{3}{8}$ in. If the fractions which the pupil meets are any more difficult than these, he should learn to reduce them to decimals and then add.

2. *Subtraction.* The objectives in subtraction are similar to those in addition and need not be further elaborated. They simply constitute the inverse cases.

3. *Multiplication.* The kinds of work that the pupil may perform are represented by such cases as finding $\frac{7}{8}$ of 10, $5 \times \frac{2}{3}$, $\frac{1}{2}$ of $\frac{3}{4}$, and $4\frac{5}{8} \times \$3.25$. At least these cover practically all the situations that arise in business. The pupil may need to find the result in such a case as $\frac{7}{8} \times 4\frac{3}{8} \times 6\frac{1}{2}$, but he will never need to know how to find a result in such a case as $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{10}{17}$. Such a case as $2\frac{7}{8} \times 4\frac{3}{4}$ may be kept in the schools for the purpose of widening the range of possible applications of mensuration, but it is a fictitious case for most people.

4. *Division.* Since the decimal is so rapidly replacing all but the simplest fractions, the objectives we give for division may possibly be obsolete in a few years. We may still need to consider, however, such cases as that of dividing a fraction by a whole number or of dividing $4\frac{1}{2}$ by $2\frac{7}{8}$, the latter being very rare. We ought to be fair with the pupil when we ask him to divide $\frac{7}{8}$ by $\frac{3}{4}$, merely showing him that the result is reasonable, but not holding it up as a very practical objective.

5. *Reduction.* The work in the reduction of fractions should be limited to such cases as that of reducing the mixed number $5\frac{2}{3}$ to the fractional form $\frac{17}{3}$ or to the decimal 5.67, of reducing $\frac{11}{3}$ to a decimal, and of reducing $\frac{1}{2}$ to $\frac{4}{8}$.

6. *Terms.* As long as they are used in business circles the terms "numerator" and "denominator" will need to be retained. We no longer need to trouble children with the least common denominator, however, because in adding or subtracting fractions

they can find the necessary denominators by inspection. For example, to add $\frac{1}{2}$ and $\frac{3}{4}$ the pupil adds $\frac{2}{4}$ and $\frac{3}{4}$.

7. *Fractional relations.* The pupil should be led to understand that 7 is $\frac{7}{8}$ of 8 and that this includes finding what fraction 7 is of 8.

8. *Ratio.* If we are asked to find the ratio of two numbers, we always divide. We may therefore teach that a ratio is a fraction, and include it here. In the formula $C = 3.14\,d$, however, the 3.14 (the ratio between C and d) is used as a multiplier, and it is so used in all similar cases in practical work. Hence we shall often improve the situation by calling attention to the fact that we frequently use a ratio as a multiplier. According to circumstances, therefore, we may have to think of ratio as a fraction, as a decimal, as a whole number, as a per cent, as a quotient, and in many practical cases as a number which we need to use as a multiplier.

9. *Proportion.* A great deal might be gained if, where the word "proportion" is introduced, it were looked upon merely as a name for an equation between two fractions. There is little gained by introducing the topic of proportion as distinct from fractions in arithmetic, although the words "ratio" and "proportion" have a legitimate place at this time.

Objectives with Decimals and Per Cents. Because decimals have rapidly replaced the common fractions in much of our work, we should be careful to lead the pupil to appreciate this change. We may then consider the following:

1. *Reading decimals.* When the pupil understands the principle which underlies the method of expressing decimals, he will have no trouble in reading or writing them.

We seldom need to read 100.003 as "one hundred and three thousandths." The phrase is too long, and the careless person is likely to confuse 100.003 with 0.103 if he hears the above stated. It is much better to teach the pupils to say "one hundred, point, o, o, three," — "point" meaning decimal point.

2. *Writing decimals.* For most people the writing of decimals is limited to three places; that is, to thousandths. We should be careful where decimals appear singly, as in "five tenths," to

write 0.5 instead of .5 in order to make sure that nothing has been forgotten or that the decimal point is not in the wrong place. If several decimals are written in a column, say for the purpose of addition, this is not necessary, for in this case the decimal point is not easily overlooked. In this connection we shall need to develop the ability to express in decimal and percent form the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{1}{8}$, and there is some value in doing the same for $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, and $\frac{1}{6}$.

3. *Kinds and extent of decimals.* Since decimals arise as a result of measurement, we need to teach the pupil how far a decimal should be carried out in a result. For example, $\frac{1}{3}$ should not be written $0.3333\frac{1}{3}$, or even 0.3333, except in unusual cases. This last form is called a *repeating decimal* or a *circulating decimal,* but the theory of these decimals is of no importance in the junior high school. The pupil needs to know that in practical measurement all results are merely approximate. They are simply accurate to the nearest unit selected, — for example, $\frac{1}{8}$ in., 0.01 in., and the like.

If a decimal is carried to hundredths of an inch (that is, to the nearest hundredth of an inch) in one measurement in a problem, we must carry it to hundredths of an inch in all other similar measurements in that problem. This is why a column of figures like

$$\begin{array}{r} 3.56 \\ 12.7 \\ \underline{.005} \end{array}$$

never occurs in actual life, although we may very likely meet with such cases as the following:

$$\begin{array}{r} 3.560 \\ 12.700 \\ \underline{0.005} \end{array}$$

A decimal like 7.56 is called a *mixed decimal,* and 0.48 a *pure decimal,* but the distinction is not an important one, and no definitions of such terms need be required.

4. *Addition and subtraction.* The pupil should be taught to add and subtract decimals which refer to actual measurements,

abstract decimals being in themselves of no importance except for drill purposes. The measurements that concern us here are those that give rise to decimals such as the most common ones of dollars and cents, automobile dials (with decimals of a mile), time-tables, and inches (where tenths are now coming into quite common use).

5. *Multiplication.* The main objective in multiplication is to teach the pupil where to put the decimal point. The best rule to use is common sense, although this should lead to the familiar rule given in the textbooks. So far as the process of multiplying is concerned there is nothing new to learn. The pupil merely places the point in accordance with what he knows about the size of the numbers with which he deals. Such procedure will help to develop the pupil's number sense and will at once lead to the common rule.

The following cases illustrate the objectives sought:

a. The ability to multiply a number expressing money by a whole number, a whole number and a fraction, or a mixed decimal like 25.8.

b. The ability to multiply 2.5 by 1.3 in order to find an area.

c. The ability to find a product like that of 15.7, 8.9, and 5.2 in order to find a volume.

d. The ability to use common sense with respect to the proper degree of accuracy in the writing of results.

e. The ability to find, in a course in general science, the number of centimeters in 9.3 in., knowing that 1 in. = 2.54 cm.

6. *Division.* The chief ability needed here is that of placing the decimal point in the right place. The ordinary needs of everyday life are covered by such cases as \$256.86 ÷ 24, \$15.72 ÷ 3.8, and 5.86 ÷ 14.5, the last being a very improbable case. The scientist might need to divide 18.78256 by 0.38754, but the average citizen will never need to do so, and in any case such long operations are usually performed on a calculating machine. We should, however, recognize that even in the elementary science of the high school there is considerable need for the division of decimals. The following cases, taken from

current texts upon the subject, illustrate the maximum degree of difficulty ordinarily found:

$50 \div 0.468$	$5.36 \div 2.07$	$1 \div 3.14$
$16 \div 15.89$	$76 \div 2.540$	$1155 \div 3.1416$

Even with these demands the work is much simpler than that which is found in the arithmetics of the older type.

7. *Per cents as decimals.* If the work in decimals is well done, percentage becomes merely a part of that topic. We would really be better off if we did not use the term "per cent" at all. Whatever we ask the pupils to do with it can be done as well by using hundredths. Business usage, however, requires the use of per cents, and hence it should be shown that 6% is merely another way of writing 0.06 and that each is only a modern way of writing $\frac{6}{100}$, which in turn is equal to $\frac{3}{50}$.

Thus the pupil should be led to see that while the applications of percentage differ from the ordinary ones of decimals chiefly because the term "per cent" is used, the mathematical principles underlying each are identical. If a pupil finds 6% of \$450, he simply multiplies \$450 by 0.06,—purely an operation in decimals.

8. *Special objectives.* In connection with a study of per cents we need particularly to secure these mechanical objectives: (1) the ability to use the fractional equivalents of $12\frac{1}{2}\%$, 25%, $33\frac{1}{3}\%$, and 50%, and possibly of $16\frac{2}{3}\%$, $37\frac{1}{2}\%$, $62\frac{1}{2}\%$, $66\frac{2}{3}\%$, 75%, and $87\frac{1}{2}\%$, although these are rarely needed; (2) the ability to understand the meaning of such expressions as $2\frac{1}{2}\%$, $\frac{3}{5}\%$, 5%, 0.55%, 100%, 125%, 1.25%, and 12.5%, and to know that $\frac{1}{2}\%$ and $\frac{1}{2}$ of 1% have the same value.

9. *Terms used.* The main objective with respect to terminology is to have the pupils understand that "per cent" means "hundredths." If they understand this, they will learn readily to write any per cent with a decimal point instead of with the symbol %. Thus 6% = six $\frac{1}{100}$, $1\% = \frac{1}{100}$, and $\frac{1}{2}\% = \frac{1}{2}$ of $\frac{1}{100}$. It is also helpful to think of 10% of a thing as a certain part of it, namely $\frac{10}{100}$ or $\frac{1}{10}$.

In order to make clear the meaning of expressions like 100%, 300%, and 125%, we proceed in the same way; that

is, $100\% = \frac{100}{100} = 1$, $300\% = \frac{300}{100} = 3$, and $125\% = \frac{125}{100} = 1.25$, or $1\frac{1}{4}$. We should also make use of measurement; thus, in an isosceles right triangle the hypotenuse is approximately 141% of either side.

Fortunately the terms "base," "percentage," "amount," and "difference" are going out of use, and only "rate" is now of any value to the pupil unless and until he takes up the study of the formula $P = BR$. In practical work these terms are replaced by such expressions as "face" (of a note), "net amount" (of a bill), "discount" (on a bill), and "premium" (on an insurance policy).

10. *Percentage cases.* The teacher will do well not to retard the work here by overemphasizing the "cases." There is only one "case" that stands out as very important for all, and this can be introduced without telling the pupils that it is Case I. If a member of the class is imagined to have bought an article marked $10 at a discount of 20%, this naturally leads to an understanding of discount. A similar approach may be made to a profitable study of commission, interest, and other applications.

The two other cases are of less importance, but are sufficiently practical to be worth teaching. They may be illustrated by the following questions:

a. $15.25 is what per cent of $350?

b. $15.25 is $5\frac{1}{2}\%$ of what number?

11. *Use of equations.* After the pupil has studied algebra the last two cases referred to in the preceding paragraph can be handled to advantage by using equations in the solution of applied problems. For example, the question "4 is 15% of what number?" may be translated into the equation $4 = 0.15\,n$, which can be solved for the value of n. All such instances of percentage can be treated in a similar way without using the terms "base," "rate," and "percentage." We have to assume, of course, a knowledge of the methods of solving such simple equations.

Objectives with Denominate Numbers. Whenever a number has a label referring to a measure, as in the case of 3 ft., it is called a *denominate number.* A *compound number* is one that is

made up of several denominate numbers of the same kind, as 4 hr. 10 min. 12 sec. Since such compound numbers were introduced to avoid the use of fractions and since today we have a much better understanding of fractions and decimals, there is now no practical need for compound numbers beyond such simple cases as 2 ft. 3 in. and 2 hr. 48 min.

Although some books still retain such compound numbers as 3 yr. 9 mo. 25 da. in connection with interest problems, the modern method of computing interest renders such numbers of little practical importance. We may safely, therefore, confine the work to a few compound numbers of two denominations involving length and weight, with possibly three denominations in the case of time. For all the rest it is advisable to use fractions and decimals.

In connection with denominate numbers we may consider the following measures:

1. *Length.* This refers particularly to such common units as the inch, foot, yard, mile, and (in rural communities) the rod. As a matter of general interest because of its frequent use in popular scientific articles, the light-year may also be mentioned, it being the distance that light travels in 1 yr. Certain stars are hundreds and even thousands of light-years away.

2. *Weight.* This means a knowledge of the ounce, pound, and ton (2000 lb.). In certain localities it is also necessary to know the long ton (2240 lb.).

3. *Liquid units.* The only units of liquid measure that we need to teach at this time are the gill, pint, quart, and gallon. In New York, for example, cream is often sold by the gill.

4. *Dry units.* As to the dry units, the quart, peck, and bushel will generally suffice, but the pint may need to be mentioned. The slight difference between the dry and liquid quart should not be emphasized.

5. *Surface units.* In square measure we need the square inch, square foot, square yard, square mile, and the acre, possibly with the square rod in rural communities.

6. *Time units.* While we need to know only the second, minute, hour, day, week, month, year, decade, and century,

the word "fortnight" (fourteen nights) should be explained by the teacher informally.

7. *Volume units.* The only units of volume that we need at this time are the cubic inch, the cubic foot, and the cubic yard, the latter being used in connection with excavations.

8. *Angle.* In the work with angles the pupil should have a knowledge of degrees, minutes, and seconds, but he should know that at the present time seconds and even minutes are not used in ordinary measurements. We are coming more and more to use decimals in our problem work. For example, we say that a certain angle contains 25.4 degrees and we write $25.4°$ instead of $25° 24'$.

The pupil should be familiar with acute, right, obtuse, and straight angles and should know what an angle of $360°$ means.

9. *Degree of accuracy in measurement.* The pupil should understand what is meant by accuracy in measurement, namely, that in most cases it means approximate and not absolute accuracy. If a pupil claims an unreasonably high degree of accuracy in an ordinary measurement, he is either ignorant or dishonest and should be shown his error.

10. *Convenient equivalents.* It is well for the pupil to have his attention called to certain approximate equivalents, such as the fact that 1 bu. contains $1\frac{1}{4}$ cu. ft.; that there are 231 cu. in. in 1 gal.; and that 1 cu. ft. of water contains $7\frac{1}{2}$ gal.; but these facts need not be memorized. There is no particular reason why any of us should remember that there are 231 cu. in. in a gallon, or the number of cubic inches in a bushel, but it is desirable that we should know where to obtain the information in the rare case of needing it.

11. *Applications.* The applications of the above measures are the practical ones that are likely to be used at home, in school, or out of doors at play or work. They include the finding of certain lengths, areas, and volumes, and the reading of gas and electric meters. It is probable that the latter represents merely an endeavor to find seemingly practical applications and is carried far beyond the reasonable needs of most pupils.

12. *Scale drawings.* The pupil should be taught not only to appreciate the use of scale drawings but also how actually to

make a drawing of his classroom to scale. He may also be asked to draw a plan of the entire school grounds, of a garden at home, of a house, or of any other familiar plane figure.

13. *Addition.* We satisfy all the needs of most people if we teach them to add such compound numbers as 14 ft. 9 in. and 8 ft. 7 in., or 7 lb. 6 oz. and 4 lb. 10 oz., even the latter being very exceptional in practical life.

14. *Subtraction.* In subtraction we may safely consider only the inverse cases of such problems as those in the preceding paragraph.

15. *Multiplication.* Since we may need to multiply 4 ft. 8 in. by 7 or to express 9×15 in. as yards and inches, we may say that the objective in multiplication will be reached if we develop the ability to multiply a compound number of two denominations by a whole number of one or two figures.

16. *Division.* The ordinary citizen may need to know how to divide 10 ft. 8 in. by 2, or even by a number of two or possibly three figures, and the objective in division may therefore be limited to such simple cases.

17. *Metric system.* Since the metric system is now used in all kinds of scientific work, the pupil should understand its general nature; he should see that it is similar to our system of money. Just as 1 mill is \$0.001, so 1 millimeter is 0.001 meter, and just as 1 cent is \$0.01, so 1 centimeter is 0.01 meter.

The question of the adoption in this country of the metric system, now used by most countries except those in which English is spoken, is not involved in this connection. It is already legal and it is used in all scientific laboratories in our colleges and high schools. It is also used by many physicians, by factories, in the jewelry industry, in the optical industry, in radio wave-lengths, and in international athletic events. It follows that the occasional use of parts of the metric system in the junior high school will be a help to the student in his subsequent work.

18. *Metric length.* The principal measure of length is the meter, and the others which we need to use are usually limited as follows:

$$1 \text{ millimeter (mm.)} = 0.001 \text{ meter (m.)},$$
$$1 \text{ centimeter (cm.)} = 0.01 \text{ meter},$$
$$1 \text{ kilometer (km.)} = 1000 \text{ meters}.$$

The meter is equivalent to 39.37 in. in our system of measures, but for ordinary use we may think of 1 m. as equivalent to about 3.3 ft. or 1.1 yd. The centimeter is equivalent to a little more than $\frac{3}{8}$ in., or about 0.4 in., as is seen from the figure on page 117, and the kilometer is equivalent to about $\frac{3}{5}$ mi., or more precisely to 0.62 mi.

The part of the ruler shown above is about 10 cm. long, and each centimeter is divided into 10 mm. If pupils do not have rulers marked in millimeters, a copy of the metric part of this ruler on a strip of cardboard or stiff paper can be made.

19. *Other metric units.* We should also emphasize the importance of knowing the gram, kilogram, and liter and should see that the pupil knows that a kilogram is equivalent to 2.2 lb. and that a liter is equivalent to 1 qt., all such equivalents being merely close approximations. There is not much chance that the metric system will suddenly replace the common system in use in this country. Such an event is not possible, it is not necessary, and it may not be desirable. If we simply teach the few units above mentioned and let them be used for scientific work and not for measuring gardens or the weight of meat, we shall naturally carry the system along with our own, and the future will determine whether or not it shall gradually replace the one now in common use.

20. *Reduction.* The reduction of measures may properly be limited to cases of two denominations, such as 3 ft. 8 in. = 44 in., $\frac{1}{2}$ yd. = 18 in., $10° 18' = 10.3°$, and so on. This limitation is due to the fact that we are not justified in teaching in the junior high school what few if any of the pupils will ever need.

Social and Civic Objectives. Whatever social objectives seem to be worth while, they should be developed only when the social status and the mental maturity of the pupils are such as

to give reasonable hope of success. It is therefore evident that we must again examine the purpose in view in teaching arithmetic, — this time with respect to the applications. This is best done by considering two questions: (1) What applications of arithmetic does the general well-educated citizen need to use commonly or to understand as matters of general information? (2) What applications of this nature is the pupil capable of understanding? These two will now be considered in detail.

Applications Generally Useful. In answer to the first of these questions a fairly complete list of essential applications can easily be made, as follows:

1. *Percentage.* Percentage merely as a form of decimals, but with emphasis upon the direct case of finding some per cent of a number. The inverse cases are of relatively little value and, although given, need correspondingly little attention.

2. *Mensuration.* Mensuration limited to those forms which people need to study for purposes of general information, and including the area of a rectangle and a triangle and the volume of a rectangular solid and a cylinder. It would be difficult to find any cases in general use that are not covered by these four. Others will naturally be taught as matters of interest, but they can hardly be required for examinations upon the essentials. This work naturally includes the use of the common and metric systems of measures to the extent mentioned on pages 115–118. Such measurements may properly make use of proportion, even though this topic is no longer extensively used in simple, common commercial problems relating to retail costs; and even the simplest notions of trigonometry may be included for purposes of general information.

3. *Application of per cents.* These include discount, simple interest for short periods, compound interest (enough to grasp the significance of the term as used in savings banks and common investments), insurance in its direct case but not in any unreal inverse cases, the significance of taxation and the general method of finding a tax rate and an individual's tax, corporation finance with reference to the meaning of stocks and dividends, investments with respect to the safety of different types and to

the opportunities for the small investor, computations of profits and losses in ordinary lines of trade, banking so far as it relates to opening an account and to the use of checks, the borrowing and lending of money, and home economics.

Such a list is easily expanded and made more specific, as is done in Chapter III; but if these topics are fairly well covered, the practical objective will be satisfactorily attained. The part that has been retained is that which has to do with our social needs, our daily relations to the community and our fellow men, and our duty to the world. Thrift, honesty, good fellowship, and general knowledge of the common life of our people, — these are the criteria by which we should select the material for instruction.

4. *Problems of the home.* There are certain items of information concerning the problems of the home which may be illustrated by the following:

a. Place of money. The pupil should appreciate the reason why money is used and should know all the common coins and bills.

b. Personal account. It is important that every boy and girl learn how to keep a personal cash account. The customary practice is simple and need not be given here. The revealing of personal accounts in school should, of course, not be required or even encouraged, and similarly for home accounts and budgets.

c. Household account. This means an enlargement of the personal account to include a statement such as the mother might keep of the receipts and expenses in the management of the home. Similar accounts might be kept in the store or on the farm.

Whatever problems are taught we need to remember that there are four tests to apply to each one, as follows: (1) Does it show originality? (2) Is it succinctly stated? (3) Is it clearly stated? (4) Does it arouse the pupil's interest?

d. Budgeting. If we are to encourage thrift generally, everyone must know how to plan ahead and estimate some sort of personal or family budget showing the probable receipts and expenses for the ensuing year and the division of income on a percentage basis.

e. Yearly inventory. This means that every person or family should take stock at least once each year to see whether he is ahead of his last year's record or behind it. Such practice ought to encourage thrift in the home. A similar practice is carried on in most stores.

f. Applied problems. The pupil should be taught to use his arithmetic constantly in solving the problems he meets in life. For the home there are problems concerning the cost of gas, electricity, telephone, and heating; the value of taking advantage of sales on clothes and food; and the relative value of buying by the piece or in quantity, and of cooking by gas or by electricity. For example, Mrs. Brown is planning to buy Jack a suit. The price of such suits as she would like to buy is $24, but she reads in the evening paper that they are to be marked down 20%. She should, therefore, be able to see how much she can save by taking advantage of the sale.

5. *Arithmetic of simple purchases and trade.* The pupil should know something about the arithmetic of the store and of the shop. In fact most of the arithmetic that the average citizen needs to know centers around that of daily purchases. The following terms should be understood:

a. Sales slips. The pupil should know that a sales slip is a memorandum of a transaction in which goods are sold at a store.

b. Bills. The meaning of a bill can best be secured by seeing a bill of goods sent out by a store. The distinction between a sales slip and a bill should be made clear.

c. Making change. Every pupil should know the common method of making change by additive subtraction. For example, if a customer buys thirty cents' worth of candy and hands the clerk a dollar bill in payment, the usual method is for the clerk to hand the customer seventy cents in change, say two quarters and two dimes, thinking as he does so "thirty and ten is forty and ten is fifty and twenty-five is seventy-five and twenty-five is one dollar," or, preferably, "thirty, forty, fifty, a dollar," laying down 10¢, 10¢, and 50¢ as he thinks the sums.

d. Invoices. The nature of an invoice as compared with a bill of goods should be understood.

e. Parcel post and express. The pupil should be familiar with the methods of shipping goods in small quantities.

f. Transportation problems. The pupil should also be familiar with the methods of shipping goods in large quantities, as by freight.

g. Receipts. The pupil should understand that a bill may be receipted, that a separate receipt may be given, and that, if payment is made by check, the canceled check is a receipt.

h. Discount. Ordinary commercial discount on bills or on sales should be clearly understood by everybody. It is also a good idea to show the value on the score of thrift of taking advantage of discounts on bills and sales.

i. Profit and loss. The pupil should understand the meaning of certain terms like "list price," "net price," "overhead," "cost of doing business," "margin," and "selling costs." He should also understand that some business houses compute the per cent of profit on the cost price as the base, whereas others, chiefly large retail stores, compute profit on the selling price. Since these practices relate to the success of large business enterprises, and therefore to the welfare of the community, the pupil should be led to understand as far as possible their general significance.

j. Commission. Everybody should know the meaning and purpose of commission and how it is computed. The pupil should see that trade could not be carried on successfully unless someone, acting as an agent, sold goods for someone else. It should be clear also that such a salesman has a legitimate claim for a part of the receipts from the sales which he makes.

k. Pay rolls. Workman and employer alike should know the basis of a pay roll, and the general plan of making one.

6. *Arithmetic of banking.* While the pupil should not be expected to understand all the details of banking practice, he should be familiar with some of the more common terms and practices involved. The chief objectives relating to this work are as follows:

a. Post-office banks. The pupil should understand the significance of the Postal Savings System, which is in effect a government savings bank.

b. Savings banks. He should know the purpose of a savings bank, how a person does business with such banks, and what compound interest means.

c. Commercial banks. The pupil should know how to deposit money in the bank and how to withdraw it. He should also know the nature and purpose of a pass book, a check, a draft, bank discount, collateral, and how money is borrowed from a bank. He should understand that the term "(bank) check" is rapidly replacing the term "(bank) draft."

d. How money is transmitted. The pupil should understand how to use checks, drafts (bank checks), and money orders in transmitting money from one place to another.

e. Building and loan associations. As stated in Chapter III, in sections of the country in which they are common, the pupil should understand the nature and purpose of such organizations as building and loan associations. In certain localities he should become familiar with the details of carrying on the business of such associations in view of his probable dealings with them later.

f. Applied problems. The pupil should be given genuine business and economic problems in the solution of which he will be obliged to use the arithmetic processes which he has learned.

7. *Arithmetic of the community.* The arithmetic which the pupil will use in connection with community interests may properly be called the arithmetic of social and civil life. His needs in this field may be illustrated by the following objectives:

a. What the community does for the citizen. In considering what the community does for the individual the pupil should study the various benefits which are derived from living in the town, city, state, and nation. Such benefits include good roads, good schools, better health, protection of life and property, and the like. The cost of these items should also be considered.

b. A person's duty to his community. Under the head of duty to his community the pupil should consider the justice of taxation of various types. The citizen's responsibility in paying his taxes cheerfully and promptly because of the services the government has rendered him should also be discussed. Methods

of collecting various kinds of taxes, duties (specific and ad valorem), and revenues should be made reasonably clear to the class.

c. Insurance. Each pupil should realize the importance of insurance from a social point of view. He should understand why insurance up to a certain limit should, if possible, be carried on the life of each working member of a community, and why there should also be insurance against fire, accident, property damage, and so on. The teacher should stimulate discussion on the need for insurance, its purposes, and the nature of the various types, such as life, fire, and liability insurance.

8. *Thrift and investments.* This subject of thrift and sound investments is one of the most important in the field of social and civic arithmetic. The following topics should receive careful attention:

a. The relation of savings to success. Although the possession of money does not mean everything, it is a great comfort for everyone to know that he has something saved for a "rainy day." Every pupil should know that at sixty years of age the majority of men are dependent upon others for support. In other words he must realize that thrift is the key to independence in old age. Here is the place to show the pupil how money grows, especially when interest is compounded. Simple graphs to show that a sum of money will double itself at 6% compound interest in a little over 12 yr. are interesting to the pupils and will help them to appreciate how "money makes money." It should be explained that compound interest can always be obtained by the simple expedient of reinvesting all interest instead of spending it.

b. How to avoid bad investments. The pupil should understand some of the danger signals of unsafe investments. For example, the pupil should be taught the need of being careful in buying certain kinds of stock that conservative banks feel to be what is usually characterized as risky.

c. Sound investments. It is, of course, perfectly evident that not all investments are bad. Pupils should be taught the fallacy of the idea that the only safe thing to do with money is to bury it or hide it. There are sound investments of many kinds.

Usually an investment paying the current rate of interest is safer than one paying an abnormally high rate. Everyone should know that when he is in doubt, he should consult a reliable banker who has no interest in the particular security under consideration. The judgment of such a man is almost certain to be far more reliable than that of one who is trying to sell the securities, however honest a salesman he may be.

d. Bonds and stocks. If a corporation sells bonds, it is really borrowing money and is giving the bonds as security for the debt. To do this it has to pledge its resources and earnings to show its ability and good faith in promising to pay back the principal of the bonds with interest. The general meaning of newspaper quotations of stocks and bonds should be understood.

Not all bonds, however, are safe investments. The pupils should know that, beginning with bonds like those issued by the government and going down the list of state, county, and municipal bonds, we come to those issued by railroads, gas and electric companies, and so on. If he is in doubt as to whether certain bonds or stocks are safe investments, the pupil should know that, again, a reputable banker's advice is the best guide.

e. How to cultivate thrift. The pupil can learn how to cultivate thrift by learning to distinguish between the necessities of life and mere luxuries, by keeping a cash account showing all income and expenditures, and by saving a little each week, no matter how small his allowance. The following table taken from a recent government booklet entitled *How Other People Get Ahead*, shows how three men divided their income:

	Miser	Spendthrift	Thrifty Man
	Per Cent	*Per Cent*	*Per Cent*
Living expenses	37	58	50
Education	1	1	10
Giving	1	1	10
Recreation	1	40	10
Savings	60	0	20

The pupil should understand the significance of thrift as shown by this table.

f. Interest. The objectives in connection with interest are to have the pupil understand the nature and purpose of this phase of arithmetic, to distinguish between simple and compound interest, and to realize the significance of the latter in relation to the cultivation of thrift. The formulas $i = prt$ and $A = p(1 + rt)$ should be understood by the pupil.

g. Promissory notes. The pupil should know when it is justifiable to borrow money and when it is not. He should know how to give a note and what such a procedure involves. Every one should be taught the danger of guaranteeing payment on a note given by some other person. Incidentally the pupil should be taught that most of the borrowing at the present time is done from banks instead of from individuals. The pupil should appreciate the fact that when a business man borrows money at a bank, a good reputation is his greatest asset. In other words, a good reputation is one of the leading objectives for all citizens.

h. Corporations. The purpose and significance of corporations in our social and civic affairs should be clearly understood. Although there are some corporations that do not deserve public respect, there are others that deserve the confidence and support of everyone. Such types of corporations should be made known in the schools and in the community.

9. *Statistics and graphs.* Everyone concedes the value of a knowledge of simple statistics and graphs. The trouble comes when we try to draw a line between essentials and nonessentials. The schools are warranted, however, in recognizing the following objectives in presenting the subject:

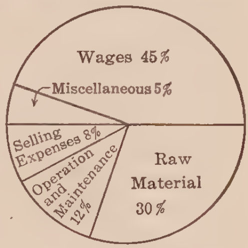
a. Methods of representing facts. The pupil should know that facts may be represented by tabular, graphic, or formula methods. The first two concern us here. The table is necessary because it gives all the details; the graph, because it pictures the facts more vividly than the table.

b. Kinds of graphs. The pupil should be familiar with the types of graphs which he will meet in his reading. These include pictorial graphs, circle graphs, bar graphs, broken-line graphs, and curve-line graphs.

c. Interpretation of graphs. The pupil must "read with" graphs, whether he has to make them or not. He must, therefore, develop the ability to "interpret" graphs, and to know when they do not tell the facts clearly, and when, as is too often the case, they are grossly misleading.

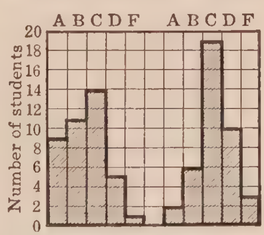
d. Use of squared paper. Squared paper ruled to centimeters and millimeters, as well as that ruled to inches and fractions of an inch, should be used in connection with all graphic work. Squared paper ruled to the metric scale is more convenient for graph work, although paper ruled to inches and tenths of an inch is satisfactory and in some cases is more easily used. All the different types of paper referred to above can usually be obtained from any bookstore that deals in school supplies.

e. Circular graph. The circular graph can best be understood by considering a single illustration. For example, this graph was drawn to analyze the distribution of the expenses of running a factory in a recent year. It shows the relative amounts in per cent spent for each purpose, and tells the story more quickly and vividly than it could be told by a table of figures. Thus we see at once that wages were nearly half of the total expenses, raw material about one third, and so on. Since circular graphs are especially effective in showing statistics given in per cents, they are often used in business.



Analysis of Factory Expenses

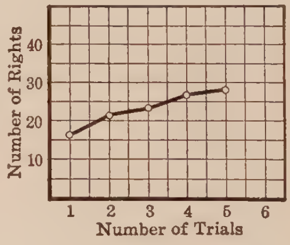
f. Bar graph. The bar graph, frequently seen in magazines and newspapers, can best be understood by considering a case relating to school work. This graph, for example, shows the number of students who obtained the marks A, B, C, D, and F in each of two sections of the same class. In this school the passing marks, beginning with the highest, were A, B, C, D, and the mark F was used for a failure.



Final Marks of Sections I and II

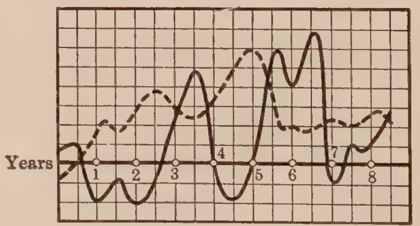
Such a graph should be discussed until each member of the class understands its construction and is able to interpret or to draw conclusions from it. In particular, the class should discuss which section had the better record, how many students received each of the various marks, and how the number of failures compared with the number of A's in each section, and the members should give the reasons for each conclusion.

g. Broken-line graph. The pupil may be encouraged to keep a record of his scores on various trials of a certain test. For example, suppose that several times during the first month of the school year John tried a drill exercise in which there were 30 examples in addition. The number of correct results that he obtained within the time limit on the first trial was 16; on the second trial, 22; on the third trial, 24; on the fourth trial, 27; and on the fifth trial, 28. John's scores may be recorded as shown in this diagram, which gives a clear picture of the progress that he made.



h. Comparison of graphs. It is interesting and often valuable to compare and to draw conclusions from the graphs of related sets of statistics.

The heavy curve of this graph shows the supply of freight cars in this country during a period of about nine years, while the dotted curve shows the general trend of the prices of standard commodities, such as food or clothing. The line marked "Years" represents the average supply of freight cars. When the heavy curve drops below that line, it indicates a shortage of cars; when it rises above, it indicates a surplus. The significance of the graph should be considered, and reasons given why one graph goes up when the other goes down.



Relation of Car Supply to Prices

A Summary of Nonessentials. For the citizen who is well informed upon the common applications of arithmetic, the essentials are those just stated. It will be necessary for him to go much farther in his own special vocation, and this he will do as part of his training in the business world. So far as the school is concerned, however, the following is a summary of topics, already discussed in detail, which may be classed as nonessentials except as they may be taught as matters of interest to the pupils but not for purposes of examination:

Inverse cases of percentage except in a few real situations such as baseball statistics; proportion except as applied to mensuration; inverse cases in interest and discount; interest upon unusual sums of money and for unusual periods of time; square root except as found from a table, this representing what is done in actual practice; partial payments except in rural communities where the subject may still be used; inverse cases in such topics as insurance; mensuration of such forms as the oblique solids, cones, pyramids, frustums, and the sphere; most work in rural measures in city schools; complicated problems relating to taxation; problems that are more easily solved by simple algebraic formulas; longitude and time except so far as it relates to standard time; tax tables; chain discounts, at least beyond two; foreign exchange beyond the simplest type; domestic exchange beyond the common methods of payment of bills at a distance; commercial paper except by casual reference; and, in general, such technicalities of banking, insurance, commerce, trade, and industry as the average citizen is not called upon to consider.

Significance of this Discussion. The purpose of the discussion thus far has been to set forth the two great aims in the teaching of arithmetic: (1) to train a child to compute accurately, with reasonable speed, in those operations that he will probably use in later life; and (2) to train him to solve types of problems that are of most value to the prospective citizen.

The discussion has opened certain necessary lines of investigation; in particular, the question of essential material, and a consideration of the nonessentials or of topics of doubtful im-

portance. It has also called attention to the fact that there are at present three sources of difficulty: (1) the tendency of state and city courses of study to perpetuate the obsolete, to which may be added the tendency to insist upon unusable and undesirable methods which may have been exploited by some enthusiastic teacher or speaker — a matter to be referred to later; (2) the tendency of printed tests to retain the obsolete or the impractical, and thus force schools to include what would otherwise be discarded; (3) the tendency of students of educational psychology to dwell upon what is in the courses of study instead of what ought to be there, thus encouraging schools to retain nonessentials; to teach as business customs those things which are not found in present practice; to misinterpret the significance of the returns from questionnaires, often in themselves hastily constructed by unskilled investigators; and to seek to change methods of instruction for some small reason which is outweighed by greater reasons which they do not understand — a subject referred to later in this work.

On the other hand, it should be clearly recognized that much progress has been made in recent years in the study of proper objectives, in the content of our arithmetics, in methods of teaching and in testing children's abilities and achievements and weaknesses, and in the knowledge of how a child learns. It is here that the best work has been done by teachers, textbook writers, makers of courses of study, and educational psychologists, and it is here that we are making distinct progress at the present time.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Outline the general range of the needs of arithmetic for the average citizen.

2. Besides those mentioned in the text, state any other topics which are occasionally found in arithmetics and which may be classed as nonessential for the well-educated citizen.

3. What and how can the teaching of taxes contribute to the better understanding of the pupil's duty to his community, to the state, and to the nation?

4. From the following standpoints discuss thrift as the key to independence:

a. What is thrift?

b. The national aspect respecting thrift.

c. How to cultivate habits of thrift.

5. What does it mean to say that our modern conception of junior-high-school mathematics will enable us to take the best practices from other countries and to introduce them by natural steps for which psychology shows us the pupil is prepared?

6. Of what value to the teacher is a knowledge of the history of certain topics in arithmetic, for example, of proportion?

7. Is it true that arithmetic is not so much computation as economics? If so, is there any justification for such a condition?

8. State the meaning of the project method of teaching the applications of arithmetic, and discuss its advantages, its disadvantages, and the probability of its having any special influence upon the teaching of the subject.

9. Name at least two reasons why the metric system has not been more generally successful in the United States.

10. How is a pupil to secure the necessary vocabulary in arithmetic without having the terms defined? What advantage will the definitions offer unless they are learned?

11. Why should we limit the operations to the necessities? A pupil enjoys an unusable computation like $2.796 \div 0.8072$ as much as he does one that he may sometime use, and why should he not have the privilege of using his time accordingly?

12. The pupil who reduces 2 mi. 18 rd. 2 yd. 1 ft. 11 in. to inches gets a considerable amount of training in the table of linear measure. Even if the case is not a very real one, why should he not have the opportunity to drill upon the table which such a problem affords?

13. How should you teach a child how to divide 34 yd. 2 ft. 8 in. by 5 yd. 1 ft. 11 in.? Discuss the value of such a piece of computation.

14. If it is stated that the sun is 92,000,000 mi. from the earth, why is not the measure given to the nearest tenth of a mile, as in automobile travel? What is the real unit of measure used in stating the above distance? Why is this unit chosen?

15. There is here shown a complex fraction; that is, a fraction having fractions for its numerator and denominator. State your reasons for or against teaching such forms in the junior high school. Do the same reasons apply to the equivalent case of $\frac{3}{7} \div \frac{4}{11}$? to any case of the division of one fraction by another?

$$\frac{\frac{3}{7}}{\frac{4}{11}}$$

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CHAPTER V

THE TEACHING OF INTUITIVE GEOMETRY

1. General Purpose in Teaching Mathematics beyond Arithmetic

General Needs of the Well-Educated American Citizen. Before considering the objectives in algebra, geometry, and trigonometry, and certain other possibilities with respect to mathematics in the junior high school, let us turn our attention for a moment to the general educational needs of our people. Why, for example, should children study any subject whatever as now taught in the schools? In particular why should we compel a pupil to learn where New Zealand is? If a man has any business there, he can in a moment look up the location on a map. Why should we care to know where Calcutta is, or what the chief industry of Japan may be, or what constitutes the chief American exports to Europe? All this information we can get from any good geography or encyclopedia if and when we need it. Even if we learn it in school, we forget most of it before we need to use it, granted that this time ever arrives.

What reason is there for requiring a pupil to learn anything about Shakespeare, or Chaucer, or Milton, or Tennyson, or any other of the great masters of English prose and verse? Was not the ancient Chinese ruler right when he decreed the burning of all the books of the empire, announcing that the past was dead and that his people should think only of the future? Is it not our duty to encourage such frequent discoveries as that literature and history should be taught by using the daily paper as a textbook and that we should "let the dead past bury its dead"? Are not the educational iconoclasts correct in saying that few read the classics of our language, whereas millions read the slangy and lurid descriptions of ball games and prize fights?

If so, should not the sporting columns be the literature of the schools, and the political news the history?

In like manner we may well question the policy of requiring science in our schools. Who cares how a coral reef is formed, what stratified rocks reveal of the story of a million years ago, or how date trees propagate their species? Enough for us to find food and clothing and automobiles and luxuries for today, letting our neighbors and the future shift for themselves.

Do we waste money in the study of music or any of the other fine arts, and is not the daubing in color done by children in the early grades a travesty upon education in an age and a land that takes pride in being intensely practical?

And so it is with every subject in our schools. We learn to write a wretched hand, while the modern letter writer uses a typewriter. Our children learn to read aloud in a halting manner, while the world in general has ceased to listen to the reading of anyone outside the church. Many schools have weekly classes in current events, while millions are nightly getting the news of the day through the radio. The child studies in dull books or is stupidly taught in a dull class the nature of a volcano, while a motion picture tells the whole story in an effective way that he will never forget.

Mathematical Needs. Nonsensical as these questions and the implications which they involve may be, they are precisely the kind of questions that we hear asked concerning algebra and every other branch of mathematics, or concerning history, language, literature, and all that helps to make an intelligent individual. The answer to any one of them applies, perhaps with slight changes, to algebra and geometry, to science and history, and to any and every important branch of human knowledge.

The reason for letting children know something about these branches of learning is that they may become well-informed American citizens. We are not training chemists in our elementary schools, nor are we seeking to make poets, historians, musicians, or mathematicians. We are simply opening the doors of knowledge; if the child wishes later to enter the domain of history and to labor there, we shall welcome him, and so for all

the arts and sciences. We should try to give to all some vision of the great departments of knowledge, so that the youth or the adult shall have some basis whereon to decide for what he is best adapted and what shall be his life's contribution to the world. For us the immediate question is, "Do algebra and geometry contribute to the end in view?"

When we come to consider mathematics in general, we find ourselves facing one of the great branches of human knowledge, just as we do in considering literature, science, or the fine arts. Anyone is happier and is more capable of bringing happiness to the world who has come to know something about the best in literature. We do not seek to make a nation of poets or of essayists. Rather would we have men and women who like to speak correctly, who do not take pleasure in vulgarity, and who have had the opportunity of knowing at least a little about a few of the classics of our language. In other words we seek to develop in our pupils what is commonly called good taste. Children who are capable of enjoying such works will have a happier future through a better use of their leisure time, and a greater opportunity to bring pleasure to others.

We may not all be Boy Scouts or Girl Scouts, or like to tramp in the woods or to observe nature, but these things make for happiness and for health. Even a little knowledge of science adds greatly to the pleasure of outdoor life and affords a purpose that makes excursions to the country seem more worth while. But for many there is a still greater reason than this. We live in an age of scientific discovery; everyone sees science on the screen, hears it in the radio, employs it in the telephone, and feels it in every line of industry. To be wholly ignorant of science is to be ignorant of modern life.

So it is with mathematics. It is one of the greatest sciences; indeed, it is the foundation of all science; it enters into every walk of life. It is desirable that children should know what its general nature is, and what people mean when they talk about per cents, algebra, trigonometry, or such things as formulas, theorems, prisms, and triangles. Such an assertion may with equal truth be made with respect to any other great branch of

human knowledge. The schools have plenty of time to open the doors to these several branches, or to merge the branches in any way that seems best, so long as they give to the pupils this vision and the opportunity to use their inherited powers to the best advantage.

The purpose, therefore, of requiring our young people to study mathematics is to give them a knowledge of what the science means, and to make it possible for them to continue further in one or more of its branches as their tastes or needs require. Let us again repeat that it is not the purpose to make mathematicians of all of them, nor even to try to make most of them capable of solving a quadratic equation or of proving the theorem of Pythagoras.

2. Purpose in Teaching Intuitive Geometry

Nature and Meaning of Intuitive Geometry. The word "geometry" originally meant "earth-measure," the subject being substantially the same as what we now, in English, call surveying. When the earth measurers flourished they developed certain rules which were also extended to include the measurement of solids. These rules were partly incorrect and unreliable, but they served as fair approximations in a period that was not very critical in matters of measurement. The idea of proving, by a train of logic, the accuracy of a rule was unknown and unthought of in very ancient times. It was only in the Greek civilization that the idea of a demonstration seems to have occurred, or at least to have been efficacious.

Intuitive geometry is somewhat like the type above mentioned. It looks at a figure and proceeds to draw certain conclusions without attempting any rigorous proof. It asserts that two vertical angles are equal because it cannot think of the possibility that they are unequal, and for the same reason it takes for granted that equal central angles determine equal arcs on the circle. The pupil at this stage looks at an isosceles triangle and says that the "base angles are equal" because, to use a familiar expression, "he feels it in his bones." The subject thus represents a stage in the childhood of the race and in the

childhood of the individual. In other words, we infer certain relations by merely looking at things.

Let it not, however, be felt that all this is unscientific. Mathematics owes its great forward steps to the intuition of its giants. Indeed, the position of pure logic may not be so firm as we usually assert, and these words of the Dutch mathematician Brower may be more significant than we ordinarily admit: "In human understanding there is no logic; in mathematics it is not certain whether all logic has validity, and it is not certain whether it can be decided, whether or not all logic has validity." For example, upon what does the *reductio ad absurdum* rest if not upon intuition?

Purpose. The purpose of intuitive geometry is, therefore, primarily to meet the necessity for visualizing and drawing the common forms that we need to use, for ordinary measuring, and for locating the positions of objects. In the realm of intuitive geometry the pupil uses the methods of most adults, — he gives freedom to his common sense in matters of useful measurements and designs. He goes still further, however, cultivating an appreciation of geometric forms in architecture, in the decorative arts, and in nature, and giving full play to his imagination. Long before he learns much about the great universe about him, he prepares the way by wandering in Lineland and in Flatland, as suggested on pages 308–314, thus coming to appreciate the possibility, now so commonly recognized in science, of a space of more dimensions than that in which we seem to exist.

Thus it comes about that intuitive geometry encourages "creative youth," and a basis is laid for an interest in the later and more formal work that many will take in science, in art, and in demonstrative mathematics.

The purpose here set forth is evidently not the same as that of demonstrative geometry, and the study of the subject is not at all dependent upon the ability to prove a proposition. The demonstrative geometries we now have are based upon those that were written for a pupil who was chronologically at least a year or two older (and mentally older still) than the one who is now studying geometry in the United States. He may not have

needed a course in the intuitive phase of the subject; today such a course is almost imperative.

Why it should be a Required Subject. Intuitive geometry should be concerned, for one thing, with the measurement of such common objects as floors, fields, and boxes, and of the silos so often seen on farms — large cylinders for green fodder. Every person needs to measure objects of various kinds, — tennis courts, cloth, flower beds, picture frames, radio aërials, and hundreds of other things. To be ignorant of how to find the area of a rectangle would be much more serious to most people than to be unable to tell where Quito and Singapore are, what the War of 1812 was about, what countries export rubber, or whether it is better to spell "criticize" with a *z* or an *s*. Intuitive geometry is, therefore, a subject to be required of all because everyone uses it.

Scope of the Work. The important question is, "How far shall intuitive geometry be extended?" Like many other subjects of study it can be elaborated far beyond the needs of most people, and like everything else it should be limited in its range by the needs and interests of the pupils.

If we consider any of the familiar objects about us, we see that we may study them in various ways. For example, we may study their composition, their color, their weight, their odor, their cost, their shape, their size, their position, and so on through a long list of properties common to all objects. For the purposes of our discussion, however, we shall consider, at least for the present, only three of the many properties that belong to every object, namely, shape, size, and position. The consideration of these three properties constitutes what we have called "intuitive geometry."

3. Geometry of Form

Shape of an Object. When we attempt to describe an object, whether it is a box, an orange, a bed, or a house, one of the first things thought of is its shape. It is in this property that primitive peoples find their greatest interest, color coming next. It was so with the race when the world was young, and it is so with

the young of the race today. That our pupils should delight in artistic forms is perfectly natural, and they would be abnormal if they did not have pleasure in coloring what they have drawn. Geometry is concerned chiefly with the forms; art is equally concerned with the color. A geometry class can easily detract from the value of the study of form by paying too much attention to color, while a class in art may lose much by considering too closely the forms of pure geometry.

Most of the forms that we need are already familiar to the pupils. Definitions avail but little at this stage in the pupils' development. We secure the best results in the way of a summary of the important forms if we look for them in nature, in daily activities, or in the schoolroom in which the class meets. It is well, therefore, to take a journey round the room and look for the forms we shall need to use.

A Journey about the Schoolroom. If we take such a journey about the schoolroom, we shall find many of the shapes that we commonly study in geometry. For example, the walls and ceiling of the room meet each other in *straight lines*, and when two such lines meet, as at a corner of the room, they meet at a *point*. The room itself is usually *rectangular*, since the straight lines which form each corner meet at *right angles* to each other, and the floor of the room is often a *rectangle* and may possibly be a *square*.

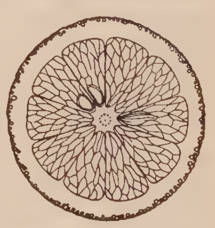
If we look at the blackboard, we see its *surface*, which is substantially a *plane surface*; that is, we may say that the surface of the blackboard is a *plane*. If the edge of a straight ruler lies on a plane, every point on the edge will lie on the plane. This is one way of testing a surface to see if it is a plane.

Very often the molding around a blackboard has a *curve surface*. On some of the furniture, perhaps a table, we shall probably find *curve lines*, which we usually speak of simply as *curves*. When we speak merely of a *line* we mean a straight line. Any part of a line is called a *line segment*.

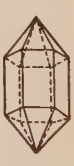
The crayon box is an illustration of the rectangular solid, and a baseball, if there is one in the room, is a good example of a sphere.

Shapes seen in Nature. Plato, the great philosopher, is reported to have said that "God eternally geometrizes," meaning that geometric forms constantly occur in nature. A little journey beyond the schoolroom will enable the pupils to see a few of the forms that nature offers. The trunks of trees are shaped like a *cylinder*, although they are often very irregular and grow smaller near the top. If we cut directly across the common type of cylinder, such as a straight, perfectly round tree trunk, we shall have a *circle*; if we cut obliquely, we shall have an *ellipse*. The planets move about the sun in elliptic paths.

Children often pour sand so that it forms a pile in the shape of a *cone*. Nature pours lava dust in much the same way, and thus we have the great cone of Fujiyama (the sacred mountain of Japan), and the cone of Vesuvius in Italy. One or both of these two mountains are probably pictured in school geographies.



If we cut crossways through an apple or an orange (which is a fairly good illustration of the *sphere* as it appears in nature), we shall find marks or lines in the cut, or *section*, which represent a geometric figure of some kind. This illustration, for example, represents a cross section of an orange.



A perfect quartz crystal, like this one, has the form of a six-sided *prism* with a *pyramid* at each end. Other crystals have different shapes; in fact, a study of crystals requires considerable knowledge of geometric forms.

Purpose of the Following Discussion. In a book like this, prepared for teachers instead of pupils, it is manifestly unnecessary to enter into the details of definitions and constructions. These are all given in any modern textbook. It is of advantage, however, to have attention called to a few important features in connection with intuitive geometry, and this will now be done.

In particular we shall consider certain facts relating to lines, angles, plane figures, and solid figures that can, in a work like this, be brought into high relief in a manner that is not feasible in a textbook for young people. We shall also discuss from the teacher's standpoint such topics as refer to drawing instruments, constructions, symmetry, similarity, and scale and pattern drawing.

In this discussion the teacher will observe that little attention is given to precise definitions. The reason is that such definitions encourage the habit of memorizing rather than of understanding. They have some place later in the study of demonstrative geometry, but at this present stage of the work illustrations and informal talks serve the purpose much better.

Suggestions respecting Surfaces, Lines, and Points. No definition of a line means much to a pupil in the junior high school; it is the illustration of a line that is important. The upper surface of the water in a glass has no thickness. Where this surface meets the glass there is a line, and this line has no thickness. In such an illustration we speak informally of a surface before we speak of a line. There is no strict logical order that need be followed in this type of work. We might even begin with solids and get good results. Take the order that makes the idea clearest. With the above illustration the pupil is prepared to understand that a geometric line has length, but it has no thickness. A wire is a rather poor illustration of a line, for it has thickness, and a chalk mark is open to the same objection.

If one line intersects (that is, cuts or crosses) another, we have a point of intersection. This is simply a position; it has no length or thickness.

A shadow on a piece of white paper may be taken as an illustration of a surface; it has length and width, but no thickness. The edge of the shadow is a line. If one shadow crosses another, the crossing of the edges illustrates a point.

It is by such simple illustrations that the pupil comes to understand the significance of the words "surface," "line," and "point."

Suggestions as to Kinds of Lines. The lines that the pupil needs to use at this time are the straight line and certain kinds of curve lines. The definition of straight line means little to a pupil who does not already have a rather good idea of what the term means. To say that a straight line is the shortest distance between two points sounds well but means little. The shortest distance from New York to London is, with the usual meaning of distance as a measurement on the earth's surface, not a straight line. Moreover, the line is not itself distance; distance is measured on a line, — it is the length of a certain line, this line being in ordinary conversation a curve, but in geometry a straight line.

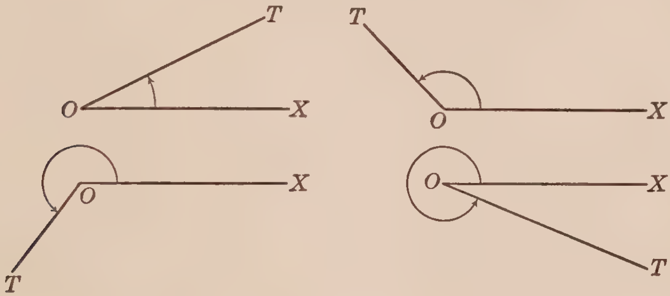
It follows that we should illustrate a straight line by speaking of the shortest path in the space with which we are familiar; of the crease made when a flat piece of paper is folded, or the path of a ray of light imagined as proceeding from a point.

A curve line may be illustrated by the path of a point on the tire of a moving automobile or by the intersection of the surface of a wave with a rock on the bank of a lake. The only curves that the pupil will use at this time are the circle and certain kinds of graphs. In due time he will learn about the ellipse (the path of the earth about the sun), of the parabola (the path of most comets), and of the hyperbola, the third of the conic sections. He will understand these if the teacher rolls a piece of paper in the form of a cone, and then cuts the cone in such ways as to show these three forms.

The expression "broken line" is of value at this time only in explaining a polygon, and even here the value is very slight. It may be omitted unless it is to be used, and in any case no definition or elaborate explanation is necessary. It should not take a minute to draw pictures of several such lines on the board and to tell the pupils that these represent broken lines.

Suggestions as to Angles. No definition of angle that is valid can be understood by the pupil at this time, but he has no difficulty in understanding a few simple illustrations. One of the best devices is a pair of shears. If opened a little the teacher can simply say, "This illustrates an angle; as I open the shears

wider the angle increases; as I close them, it decreases. Here is a right angle; as I close them a little we have an acute angle; if I open them beyond a right angle, we have an obtuse angle." After such an introduction the class is prepared for a statement about the number of degrees in a right angle; for a picture of a



straight angle (180°); and for one of an angle of more than 180°. It is also prepared for such formal statements as this:

"If a straight line, as *OX* in any of the drawings above, rotates in a plane about a fixed point, as *O*, in the direction indicated by the arrowheads until it reaches the position *OT*, it is said to turn through the angle *XOT*." Thus the size of an angle depends upon the amount of turning made by a line rotating in a plane about a fixed point. It will interest and instruct the class to know that, in the case of a screw, the

Right Angle **Acute Angle** **Obtuse Angle**

screwdriver may not only turn once around (360°), but as many times as we wish. We then say that it turns through more than 360°, and hence that an angle may, considered in this way, exceed 360°, or four right angles.

The pupils should be asked to find examples of all the above angles in the classroom. For example, let them see that the

hands of a clock form a right angle at three o'clock, an acute angle at five minutes after three, and an obtuse angle at five minutes before three, as shown on the preceding page.

They should also acquire the habit of properly reading angles. Thus, in the figure shown below, the angle designated by the arrow may be called angle O or angle AOB, — written $\angle O$ and $\angle AOB$, the vertex letter being placed in the middle. In general, it is better to read angles counterclockwise (that is, counter to the way in which the hands of a clock move); thus, in this figure, $\angle AOB$ is the one marked with the arrow, while $\angle BOA$ is the smaller angle. The rule is of no importance so long as we speak of angles less than 180°, but beyond that size it is convenient to follow the practice suggested.

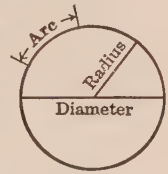
Suggestions as to Plane Figures. We may have an unlimited number of shapes of figures. A straight line is a figure and so is an angle, a sphere, a cube, or a circle. Most pictures, like photographs, maps, drawings, and paintings, are on flat surfaces, and hence are spoken of as plane figures. This is one reason why plane figures occupy so much attention in geometry.

On some accounts it is unfortunate that we use such difficult names for very simple figures. Many people like to use such names; they think that they will be thought to be more learned, — and in one way this is true. It is a good plan to tell a class something about the origin of such long words as "rectilinear" (straight-lined), "triangle" (three-angle), and "quadrilateral" (four-sided). Such rectilinear figures as the following may then be explained informally:

Triangle Quadrilateral Square Rectangle Parallelogram

Of the curvilinear figures already mentioned, the one that the pupils will use most often is the circle. Teachers should understand that mathematicians define the circle to be the line,

not the part of the plane inclosed by it. The circumference is a number, — the number of units of length in this line. The pupils will be interested in informal statements as to the origin of the word "arc" and its connection with "arch," of the meaning of "diameter" (through-measure), and of the connection of "radius" of a circle with a certain bone in the forearm. Similarly, the origin of the word "circumference" (around-carrying) and the connection of "circle" with "circus" and "circulate" will add much interest to what may easily lapse into rather stupid formalism. A few words as to the importance of the circle will also add much to the interest shown by a class. If we should be able to abolish all circles, we should have no wheels, no pulleys, no gears, no automobiles, and nothing that revolves. The world could get along without money, — it did so for thousands of years; it could exist without airplanes or silk or apples or hats; but it could not exist in its present form without circles.



Suggestions as to Triangles. In teaching the class about the simplest of the common polygons, the triangle, it is desirable to break with tradition and indulge in informal talks as in the case of the circle. The isosceles triangle means much more if the class is told that "isosceles" means "equal-legged," and equilateral is not easily forgotten if its meaning (equal-sided) is explained. The pupil comes then to see that "lateral" means something connected with "side," and he is quite prepared to see that "quadrilateral" means quadri-sided and to guess that "quad" (connected with "quart" and "quarter") means "four."

The standard forms of the triangle, with respect to sides, are the isosceles and equilateral; there is hardly any worthy reason for continuing the use of the word "scalene." With respect to angles the standard forms are the acute, obtuse, equiangular, and right, the general term "oblique triangle" being sometimes applied to the first three but being quite unnecessary. All these types are shown in any standard series of textbooks on junior-high-school mathematics and are so well known that we need not illustrate them.

Such features as the "base" and the "vertex" of a triangle are best explained informally, without any special definition.

Quadrilaterals. The pupil should know the most important kinds of quadrilaterals, as here shown:

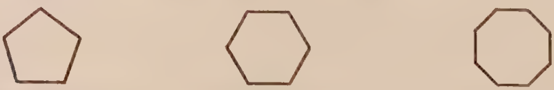
Rectangle **Square** **Parallelogram** **Trapezoid**

Suggestions as to Polygons. The class will be interested in the origin of such a word as "polygon." The prefix "poly" is from a Greek word meaning "many," and "gon" is from one meaning "angle." A triangle is one form of a polygon, and if we had preserved more nearly the Greek form we would have called it a "trigon."

Such terms as "sides," "vertices," "perimeter," and "base" are best understood if taken up at the board without any great expenditure of time. Aside from "perimeter" (around-measure), they will all be known or inferred before they are met with in the study of the subject. The meaning of a regular polygon is easily understood from an informal discussion of the figure. As to polygons of more than four sides, the following are the only ones that the pupils need to know at this time:

Pentagon **Hexagon** **Octagon** **Decagon**

The following figures show the important regular polygons:



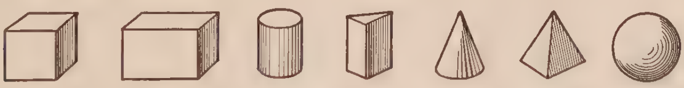
Regular Pentagon **Regular Hexagon** **Regular Octagon**

The decagon shown in the upper group is a regular decagon. Regular polygons, of course, also include equilateral triangles and squares, but in speaking of such figures we usually refer to them by their special names.

Suggestions as to Solid Figures. In the intuitive stage of geometry there should be no dividing line between the study of plane figures and of solids. In fact, there is a good argument in favor of beginning with the latter as being the more tangible. It must be borne in mind, however, that the child has already done this in his work with building blocks, balls, marbles, and cylindrical objects. What the school does is to crystallize the knowledge that he has already acquired, and to lead him to see more clearly what he has seen vaguely and with no definite purpose in view.

The one new thing that he needs to learn is that a geometric solid is considered merely as the space that is occupied by some one of the solids that he has already come to know. It may be illustrated by thinking of a cube covered thickly over with wet clay, and then of the clay as dried and sawed in two parts so that the cube can be removed. The vacant space that the cube occupied is a *geometric solid*, so called to distinguish it from physical solids that we can feel. This being understood, the pupil sees that it is possible to think of lines or planes as passing through the cube, or of a sphere as being placed inside the solid.

The following are the familiar solids commonly used in intuitive geometry:



No formal definitions of such solids are advisable at this time, and it is not necessary that all these figures should be studied. With respect to measurement, the rectangular solid and the cylinder are by far the most important. The name and characteristics of each of the above solids should be stated briefly and illustrations of each as found in nature or in building should be given by pupils, but little attention need be paid in the junior high school to the measurement of any except the two specified.

Suggestions as to Drawing Instruments. Although figures can always be drawn freehand, the results are often very unsatisfactory. This is not only true of the work of our pupils, whom

we frequently judge by the neatness of their drawings, but it is true of our own work as well. If we are ourselves unable to make satisfactory figures freehand, the pupils cannot be held to a very high degree of perfection.

If circumstances permitted us to have a large supply of instruments on hand, so that each member of the class could have what he needed, we could have some interesting laboratory work that would produce good results. As it is, we must do the best we can with a limited number of tools.

With respect to the straight line, most pupils are supplied with rulers. Since the metric measures are now used in all scientific laboratories, it is desirable that pupils should become familiar with the basal units, and hence it is convenient to have rulers with the inch scale on one side and the metric one on the other. It is also helpful to have one scale where the inches are marked in tenths. If rulers having the metric scale cannot be procured, the common one, scaled to inches and eighths, will have to suffice.

Since most of us cannot draw a very accurate circle freehand, it is necessary to have a pair of compasses. It is possible to procure these very cheaply and yet made with sufficient care to give satisfactory results. It is not necessary for a school to supply blackboard compasses, however; for a piece of string will answer quite as well.

The freehand drawing of angles of definite size is even more difficult, and so it is very desirable that the pupils should be supplied with protractors. The inexpensive paper ones are satisfactory for a time, but the celluloid or brass ones are much better.

Draftsman's triangles are very convenient and T-squares have some value in the work, but each is a luxury rather than a necessity. The ruler and compasses are necessary, the protractor is desirable, and the other two may be classed as relatively unimportant. The draftsman's triangle may, if desired, be cut out of cardboard.

Desirable Constructions. In connection with the careful making of geometric figures the most important question is to decide

upon the constructions with ruler and compasses that are most worth while to people generally. For example, is it worth the effort to know how to construct one line perpendicular to another? Do we need to do this in any plans of buildings or other drawings that we may be making? If so, should we do this with ruler and compasses or with either a draftsman's triangle or a carpenter's square? Similarly, is it worth while to know how to bisect an angle or to draw one line parallel to another? It is the consideration of such questions that leads us to see clearly the objectives in the geometry of shape.

Fundamental Constructions. Having raised the general question, we shall now consider the constructions which may be looked upon as fundamental in the mathematics course of the junior high school. These constructions, which may be given as formal problems or as exercises, are as follows:

1. At a given point on a given line construct a perpendicular to the line.
2. From a given point not on a given line construct a perpendicular to the line.
3. Bisect a given line segment.
4. At a given point on a given line construct a line which shall make with the given line an angle equal to a given angle.
5. Through a given point construct a line parallel to a given line.
6. Bisect a given angle.
7. Construct an equilateral triangle having a given side.
8. Construct a triangle having its sides respectively equal to three given line segments.
9. On a given base construct an isosceles triangle having its other sides each equal to a given line segment.
10. Divide a given line segment into any number of equal parts.

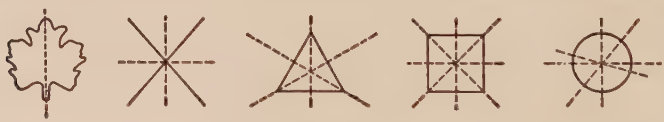
Symmetry. The pupils should be encouraged to see mathematics in all nature. In particular they should see that nature delights in making two parts of many objects so nearly alike that one seems to balance the other. For example, one wing

of this butterfly is like the reflection of the other in a mirror standing midway between them; one is the picture of the other reversed.

The pupils should be told, if necessary, that this element of beauty is called *symmetry*, and they should look for it in the human face, in the bodies of many animals, in leaves, in fruits, and in crystals of various kinds.

Axis of Symmetry. If we should fold this picture of a snow crystal along a horizontal line through the center, the lines of one half would coincide with those of the other half. The line on which we fold such a figure is called the *axis of symmetry*.

As shown by the dotted lines in the figures below, a figure may have more than one axis of symmetry. The circle at the right, for example, has as many axes as we wish to take. It may interest the pupil to find the number of axes of symmetry that there are in the snow crystal above.



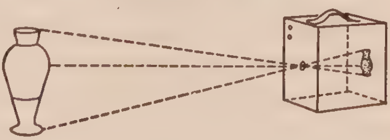
Center of Symmetry. A figure may also have a *center of symmetry*; that is, a point about which it may turn through 180° and then coincide with its original position, or a center such that it bisects all lines through the figure terminated by its boundary. In the figure here shown, O is the center of symmetry.

The pupil should find how many of the figures above shown have axial symmetry, how many have central symmetry, and how many have both. He should also be encouraged to look for the beauties of geometry in the world about him, particularly as seen in the symmetry of leaves, of the cross sections of fruits, and in various crystals, flowers, and animal forms.

Plane of Symmetry. Instead of an axis of symmetry, solid figures may have a *plane of symmetry.* This is easily illustrated by cutting an apple or an orange in halves, the knife passing through the stem and the blossom end. Other familiar illustrations of symmetric solids are found in the cube, the cone, the cylinder, and the sphere. While it is not essential to discuss the fact at any length, it will interest a class to observe that a cube has several planes of symmetry, and that we may also speak of it as having an axis and a center of symmetry, which is true of the sphere as well. A mirror is a good illustration of a plane of symmetry, the reflection being symmetric to the object reflected and being imagined to be itself a solid. In this case we would have two symmetric bodies, like a pair of gloves or like our two hands.

Similarity. If the pupil takes two maps of his state, one drawn to a larger scale than the other, we say that the maps are similar; that is, they have the same shape but, in this case, not the same size. Likewise, if we have a small photograph enlarged, the original and its enlargement are similar. One of the most familiar examples of similar figures is seen in a motion picture, each figure on the screen being similar to the one on the reel, although it is very much larger. Similar figures may be of the same size, but this is only a special case; the size is not essential; it is the shape that determines the similarity.

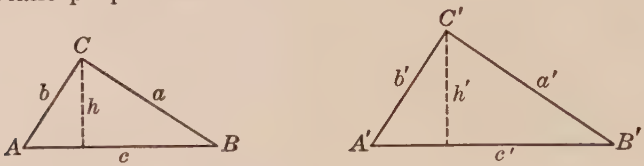
The camera affords an interesting illustration of similar figures. When we take a photograph, the rays of light pass through the lens as shown in this figure, striking the film or plate in the back of the camera. Lines in the object are reproduced on the film in



the same relative positions, although the figure is inverted. No angle is changed in size, but the lines are all reduced in the same ratio.

Mathematics of Similar Figures. If, in two triangles of the same shape, we represent the sides by the small letters corresponding to the capital letters at the opposite vertices, and

represent the heights by h and h' (read "*h prime*") respectively, we can lead the pupil to discover by experiment the following important properties:



1. *The corresponding angles are equal.*

That is, $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.

2. *The corresponding lines are in proportion.*

This means to the pupil, as he knows from his earlier work, that if a is $\frac{2}{3}$ as long as a', then b is $\frac{2}{3}$ as long as b', and so on. That is, the ratio of a to a', which means a divided by a', is the same as the ratio of b to b'.

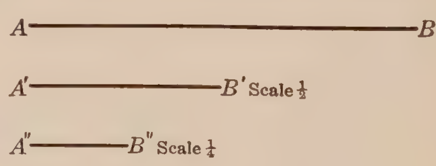
This may be expressed by using proportion, which is best thought of as a statement of equality between ratios, as follows:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{h}{h'}.$$

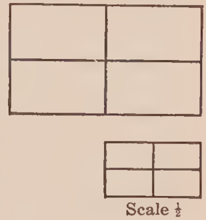
The above expression is usually read "a is to a' as b is to b' as c is to c'" and so on, but we often say simply "a, a', b, and b' (and so on) are in proportion."

Drawing to Scale. As noted on the preceding page, when a camera makes a picture it reduces in the same ratio all the lines of the object photographed, keeping all the angles the same size as in the original. Similarly, when a pupil makes an accurate drawing of an object, he must reduce or enlarge all lines in the same ratio, keeping all angles as in the object as he sees it. Such a drawing is called a *scale drawing*.

For example, in this figure, $A'B'$ shows the line AB drawn to the scale $\frac{1}{2}$; that is, $A'B' = \frac{1}{2} AB$. Similarly, $A''B''$ represents AB drawn to the scale $\frac{1}{4}$, or the line $A'B'$ drawn to the scale $\frac{1}{2}$.

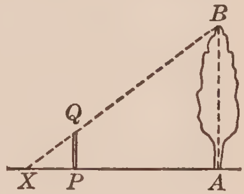


In the case of these two rectangles, the second figure shows the first drawn to the scale $\frac{1}{2}$; that is, every line is half as long and the angles remain unchanged. The expression "scale $\frac{1}{2}$" refers to the lines, not to the area.

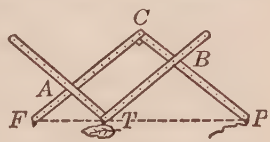


Scale drawing is, therefore, merely a general case of map drawing, a subject with which the pupil is already somewhat familiar and to which scale drawing should be related.

The pupil should be led to see that he can find the height of certain objects by scale drawings about as well as he can later find it by means of ratios. For example, this figure shows how a boy found the height of a tree. He set a 16-foot pole upright at P, and found a point X from which he could just see the top of the tree over the pole. By measuring he found that AP and PX were 24 ft. and 8 ft. long respectively. Then by making a scale drawing and by measuring AB, knowing the scale used, he was able to find the required height.



The Pantograph. It will add to the interest in similar figures if a pantograph can be shown to the class. The instrument is here illustrated and is extensively used in enlarging or reducing plans or designs. The bars of the instrument, hinged at C and T, are pivoted at A and B, and the point F is kept fixed. As the tracing point T is moved over the outline of a design, the pencil at P draws an enlargement similar to the design. It is not difficult to make a pantograph, particularly if there is in the school a class in manual training.



Patterns and Designs. While such instruments as the T-square, draftsman's triangle, and pantograph are not available in all schools, rulers and compasses are usually in the hands of the pupils in intuitive geometry. These can be used to advantage in the study of pattern work and designs as suggested

in all the better class of textbooks. Patterns for linoleum, oilcloth, leaded glass, church windows, and decorations in general afford an opportunity for artistic design, a study of geometric forms, and manual training, all of which is valuable. There is, however, the danger of carrying this kind of work beyond the point of adequate returns, especially if too much attention is paid to color work and to material produced simply for the purpose of exhibition and with little or no mathematical value.

4. Geometry of Size

Measurement. There are many things that most of us should like to measure if we could. For example, we should all like to measure our own intelligence, if we could keep the result to ourselves, and we should like to measure precisely the distance to the most remote star that our greatest telescope reveals. These measurements being impossible for us, the question arises as to what we are able to do in this part of mathematics. The answer manifestly is that we are able to measure certain lengths, including such particular ones as height, depth, width, and ordinary distances in general; areas, including at least the measurement of the surfaces of such common figures as the rectangle and triangle; and volumes, especially of rectangular and cylindric solids, and including what we call the capacity of tanks.

In all such measurements it must be clearly understood that our results are merely approximate. We usually speak of the distance between two cities to the nearest mile, of the distance to the sun to the nearest half million miles, of the length of a desk to the nearest eighth of an inch, of the distance between two screw threads, where great accuracy is required, to the nearest 0.001 in., and so on; but we never have absolute accuracy. We measure approximately, we compute with absolute accuracy. It should be observed, however, that the result of any computation based upon a measurement can never be more nearly accurate than the measurement itself. For example, the circumference of a circle is about 3.14159265+ times the diameter, but if we know the diameter only to the nearest eighth

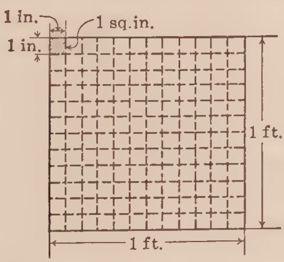
of an inch, we can find the circumference only to the nearest eighth of an inch, however accurate may be our multiplication.

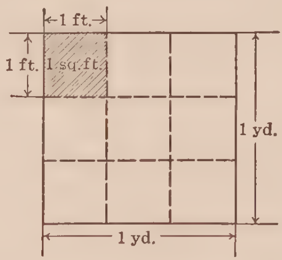
Length. In the practical measuring of lengths, fractional parts of an inch are usually expressed with the denominators 2, 4, 8, 16, 32, or 64, and with growing frequency as decimals, particularly in machine-shop work, where measurements are often made to the nearest thousandth of an inch (0.001 in.).

If we use the metric system, as is done in all scientific laboratories, we use the millimeter or the centimeter in place of the inch and its fractional parts, the meter in place of the foot or yard, and the kilometer in place of the mile. The meter is about 1.1 yd., and the kilometer is about 0.62 mi.

The approximate nature of all measurements of length can be made evident to a class by having the members, working in pairs, measure the length of the walk in front of the school, carrying their work to the nearest $\frac{1}{8}$ in. If they do this and do not compare their work until it is finished, it is quite probable that no two pairs will have the same result.

Area. In finding the areas of surfaces the square inch, square foot, and square yard, with decimal subdivisions or with the simplest fractions, are the units generally used in such work.





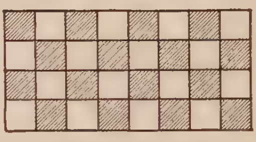
A square inch (sq. in.) is the area of a square that is one inch (1 in.) on a side; a square foot (sq. ft.) is the area of a square that is one foot (1 ft.) on a side; and a square yard (sq. yd.) is the area of a square that is one yard (1 yd.) on a side.

From the above figures, which are, of course, much reduced, it will be seen that there are 144 sq. in. in 1 sq. ft. and 9 sq. ft. in 1 sq. yd.

The pupils should understand that an inch square is a square that is 1 in. on a side, that a square inch is the area of this square, and that a triangle or any other flat figure may have an area of 1 sq. in. In other words, an inch square is a square figure, but a square inch may be the area of any flat surface, whatever the shape. The distinction is practically of slight importance, but the terms should be used properly.

The pupil should understand that there are other units for measuring areas, such as the square rod (sq. rd.) and acre (A.). These two are of greater value in rural communities than in cities. Pupils should be given practice in visualizing the acre, a piece of land having approximately this area being shown to them.

Area of a Rectangle. If the floor of a rectangular corridor 8 ft. long and 4 ft. wide is made of marble squares 1 ft. on a side, as shown in this figure, there are evidently 8 squares in each row, and there are 4 rows of these squares. Then the number of square feet in the floor is 8×4, or 32; that is, the area of the floor is 32 sq. ft.



In writing this solution it is not necessary to have

$$8 \times 4 \times 1 \text{ sq. ft.} = 32 \text{ sq. ft.},$$

or

$$8 \times 4 \text{ sq. ft.} = 32 \text{ sq. ft.};$$

it is sufficient simply to say,

$$8 \times 4 = 32,$$

and hence there are 32 sq. ft. in the area of the floor.

Since any rectangle can be divided into squares in this way, the pupil can easily be led to see that the total number of squares is the product of the number in each row multiplied by the number of rows; that is,

The area of a rectangle is the product of the length and width.

Formula. The time has now arrived for telling the pupils that they are ready for a little mathematical shorthand. They should be told that they have long known something about this labor-saving device, for they write + for "plus," × for "times" or "multiplied by," — for "minus," ÷ for "divided by," as well as = for "equals" or "is equal to."

This shorthand is now extended so that, instead of writing

Area equals length times width,

we can use initial letters and write

$$A = l \times w,$$

and can abbreviate this still more by simply writing

$$A = lw,$$

it being understood that two letters written side by side, like lw, shall be taken to mean the product of the numbers they represent.

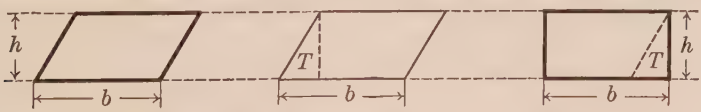
Evaluating. The next step is to find the value of, or evaluate, A for various values of l and w. For example, to evaluate A, or lw, when l represents 6 and w represents 4, or, as we say, when $l = 6$ and $w = 4$, we have

$$A = lw = 6 \times 4 = 24.$$

If l and w are expressed in feet, then A will be expressed in square feet, and similarly for other units.

In the same way the class should be taught the significance of, and method of evaluating, a^2 and a^3.

Area of a Parallelogram. While it is practically of much less importance to find the area of a general parallelogram than to find the area of the special kind that we call a rectangle, it is a simple matter to do it and it has considerable interest for most pupils. Ask them first to cut a paper or cardboard parallelogram like the one in the left-hand figure here shown.



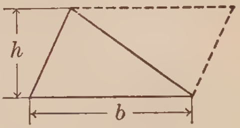
If from the left of the parallelogram they then cut off a right triangle T, as shown in the second figure, and place it at the right, as shown in the third figure, the result is a rectangle with the same height and a base equal to that of the parallelogram. That is, they learn experimentally that

The area of a parallelogram is the product of the base and height.

Expressing this relation in the more convenient algebraic shorthand, we have the formula

$$A = bh.$$

Area of a Triangle. It is easy for pupils to see that any triangle may be considered as half the parallelogram formed by drawing lines parallel to the base and to one side, as here shown, and thus to see that

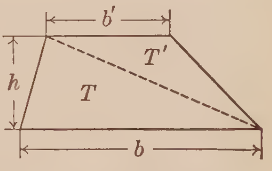


The area of a triangle is half the product of the base and height.

Using algebraic shorthand, we thus have the formula

$$A = \tfrac{1}{2}\, bh.$$

Area of a Trapezoid. Less important in actual practice in measuring, but even more interesting in other respects, is the case of the trapezoid. By drawing either diagonal, the pupil divides the figure into two triangles T and T', as here shown.



He can then see that $\triangle T$ has the base b and the height h, and that $\triangle T'$ may be considered as having the base b' and the height h.

That is, although $\triangle T'$ does not appear to rest on any particular side, he takes b' as the base. Indeed, he can always consider any side of a triangle as the base, since the figure can be turned so that it will appear to rest on this side if it is so desired.

Considering the areas of the two triangles, he can now see that

$$\triangle T = \tfrac{1}{2}\, bh,$$

and

$$\triangle T' = \tfrac{1}{2}\, b'h.$$

We might stop with this, for in practice we could find the area of each of these triangles and then find the area of the trapezoid by adding these results.

It is, however, convenient to have a single formula, and so we add these equals and obtain

$$A = \tfrac{1}{2}\, bh + \tfrac{1}{2}\, b'h.$$

Here again we might stop, for we have a formula for the area. The teacher can, however, by such a simple case as

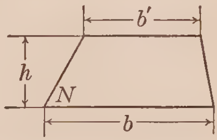
$$2 \text{ tens} + 3 \text{ tens} = 5 \text{ tens},$$

show that
$$A = \tfrac{1}{2} h(b + b').$$

Expressed in words, we have the following:

The area of a trapezoid is the product of half the height and the sum of the bases.

The teacher now has an excellent opportunity to lead the pupils to a new phase of mathematics, — the generalization of a figure and formula. Considering the expression
$$A = \tfrac{1}{2} h(b + b')$$



and the figure here shown, we see that

1. If $b = b'$, we have the formula for the area of a parallelogram.
2. If $b' = 0$, we have the formula for the area of a triangle.
3. If $b = b'$ and N contains 90°, we have the formula for the area of a rectangle.
4. In 3, if b also equals h, we have the formula for the area of a square.

The pupil should also see that the area of any rectilinear plane figure can be found by dividing the figure into triangles or, more conveniently in some cases, into triangles and trapezoids.

Measurement of a Circle. For most people a knowledge of the measurement of the circle is much less important than that of the rectangle or the triangle. If, however, we wish to know how to obtain a formula for the volume of a cylinder, such as a cylindric can, pipe, or water tank, we must first know how to measure a circle.

Let us take the simple case of finding a formula for the circumference. If we allow the pupils to measure the circumference (C) and diameter (d) of each of several circles and then in each case to divide the circumference by the diameter, they will find that each quotient is a little more than 3.1. If the measurements are carefully made, the quotient will be found to be approximately $3\frac{1}{7}$; that is, $C \div d = 3\frac{1}{7}$, approximately.

By higher mathematics it is shown that a closer approximation is between 3.1415 and 3.1416. This ratio, which cannot be exactly expressed as a decimal, is designated by the Greek letter π, called "pī." That is,

$$\frac{C}{d} = \pi;$$

and since the diameter (d) is twice the radius (r), we have

$$\frac{C}{2r} = \pi.$$

In arithmetic we know that if

$$\tfrac{6}{3} = 2,$$

it follows that $$6 = 2 \times 3.$$

Similarly here, we see that

$$C = \pi d$$

and that $$C = \pi \times 2r, \quad \text{or} \quad C = 2\pi r.$$

For practical purposes in computing, the pupil should take $3\frac{1}{7}$ or $\frac{22}{7}$ as the value of π, this being sufficiently accurate for his present purposes.

For example, if a large oil tank lies so that we can easily measure its diameter but cannot run a line about it so as to measure the circumference, we can compute the latter by the formula $$C = \pi d; \text{ that is, } C = \tfrac{22}{7}d.$$

Area of a Circle. The finding of the circumference is not so important, however, as the finding of the area.

As shown below, a circle can be separated into approximate triangles in which the height is the radius, and the sum of the bases is the circumference.

When the circle is thus spread out the pupils can see that if the sectors were exact triangles, the area (A) of the circle would be half the product of the radius (r) and the circum-

ference (C). They may now be told that in higher mathematics it is proved that this is the exact area of any circle; that is, that

$$A = \tfrac{1}{2}\, rC.$$

For practical measuring, however, this can be simplified if we put $2\,\pi r$ in place of C. We then have

$$A = \tfrac{1}{2}\, r \times 2\,\pi r, \quad \text{or} \quad A = \tfrac{1}{2} \times 2\,\pi \times r \times r;$$

whence

$$A = \pi r^2.$$

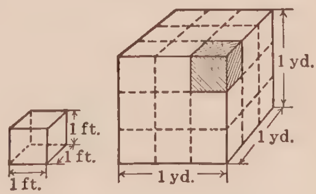
For example, to find the area of a circle in which r is found by measurement to be 10.0 in., to the nearest tenth of an inch, we have

$$A = \tfrac{22}{7} \times 10 \times 10 = 314\tfrac{2}{7};$$

that is, the area is $314\frac{2}{7}$ sq. in.

Since $\frac{22}{7}$ is only an approximate value of π, and since the radius is known only to the nearest tenth of an inch, the result is only approximate. It may be given as 314 sq. in. (to the nearest unit), or 314.3 sq. in. (to the nearest tenth).

Volumes. In measuring volumes the cubic inch, the cubic foot, and the cubic yard, with their decimal subdivisions or with the simplest common fractions, are the units generally used. The pupil should be reminded that a cubic inch (cu. in.) is the volume of a cube that is 1 in. on an edge; a cubic foot (cu. ft.) is the volume of a cube that is 1 ft. on an edge; and a cubic yard (cu. yd.) is the volume of a cube that is 1 yd. on an edge.

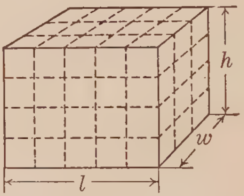


The figure at the left represents 1 cu. ft., and the figure at the right represents 1 cu. yd., the shaded block in the corner representing 1 cu. ft. By counting we can see that 1 cu. yd. contains 27 cu. ft. A large figure drawn on the board can be used in the same way to show that 1 cu. ft. contains 1728 cu. in.

Similarly, in the metric system a cubic millimeter (cu. mm.) is the volume of a cube that is 1 mm. on an edge; a cubic centimeter (cu. cm., but more often written simply cc. or ccm.) is the volume of a cube that is 1 cm. on an edge; and so on.

The only solids of which most people need to find the volumes are the rectangular solid and the cylinder. Few of us ever need to know the volume of a cone, a pyramid, or even a prism; and if we do, we can easily find the formula in a geometry or an encyclopedia. We may therefore, in the junior high school, confine our attention largely if not wholly to these two types.

Volume of a Rectangular Solid. If this figure is taken to represent a rectangular solid 5 in. long, 3 in. wide, and 4 in. high the pupil can see by counting that it takes 5×3 cubes to make the bottom layer, and that it takes four of these layers to make the solid. He can thus see that there are $5 \times 3 \times 4$ cubes in the solid, and that its volume is 60 cu. in. That is, for all such cases,



The volume of a rectangular solid is the product of the length, width, and height.

Expressed in algebraic shorthand, this becomes the formula

$$V = lwh.$$

If all the edges of the solid are equal to e, the figure becomes a cube, and the formula becomes

$$V = e^3.$$

Cylinder. Everyone is familiar with the common form of cylinder, as exemplified by a water pipe, a gas pipe, a piece of wire, or the trunk of a tree, even though none of these are really perfect cylinders.

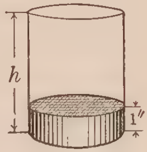
In the junior high school we should consider only the common form of cylinder, in which the bases, the two parallel faces, are circles.

In such a cylinder we may speak of the radius of the base as the *radius of the cylinder.*

Volume of a Cylinder. Suppose that we have a wooden cylinder 5 in. high and that the base has an area of 10 sq. in. (which will, in actual measurement, be only approximately exact). As suggested by the figure on the next page, we might cut off a section 1 in. high, which would give us a cylinder with a volume of

10 cu. in. Since we could evidently make five of these 1-inch sections, the volume of the original cylinder is 5×10, or 50 cu. in.; that is, we can show the pupils by this discussion that in such cases

The volume of a cylinder is the product of the area of the base and the height, $V = Bh.$



Since, in the common type of cylinder, $B = \pi r^2$, we may show the pupil how to write this formula in the more convenient form

$$V = \pi r^2 h.$$

For example, a water main has an inside diameter of 14 in. If the main is completely filled, we can tell how much water is contained in 500 ft. of length as follows:

Since the diameter is 14 in., the radius is 7 in., or $\frac{7}{12}$ ft., and so

$$V=\pi r^2h=\frac{\overset{11}{\cancel{22}}}{\cancel{7}}\times\frac{\cancel{7}}{\underset{6}{\cancel{12}}}\times\frac{7}{\underset{3}{\cancel{12}}}\times\overset{125}{\cancel{500}}=\frac{9625}{18}=534.72\cdots;$$

that is, to the nearest 0.1 cu. ft., the main contains 534.7 cu. ft. of water.

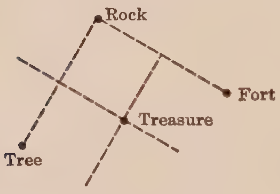
5. Geometry of Position

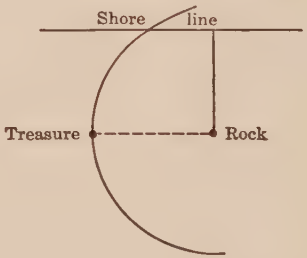
Position. The geometry of position has many more interesting applications and is much more important than might at first be thought. For example, in so simple a thing as addressing a letter to some person, unless the address gives the street and number, the place, and the state in which the person lives, the postal service will not be able to deliver the letter promptly.

To take another familiar illustration, we locate places on a map and on the earth's surface by giving their latitude and longitude, and in certain cities it is the custom to locate houses by giving the number of the street and the number of the particular house on that street. The topic can also be related to the laying out of tennis courts, football fields, and baseball fields. Until rather recently the schools have emphasized the geometry of size but have unduly neglected the equally important and even more interesting geometry of shape and position.

How to Begin this Work. Many interesting stories have been told about buried treasure in the days when there were no safe-deposit vaults for protecting valuables and no banks for depositing money. In such cases it was necessary to have some convenient method of remembering where the treasure was buried.

An interesting way to begin the study of position is, therefore, to start with a story of buried treasure similar to that in Poe's tale "The Gold Bug." For example, we might tell a story about a pirate captain who took his crew off to some desert island, buried their treasure, and then killed all or nearly all of his men. The pupils should then be asked to suggest ways in which the captain might have remembered where he buried the treasure, so that he could return later and be sure to find it easily. We might even let the pupils play the part of the captain and bury the treasure with different ideas for locating it. The diagrams below will suggest some of the possibilities.

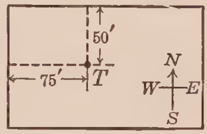




The pupils may also play an outdoor game in which each of two groups elects a captain and each side then takes turns in hiding their captain. The hiding side then gives to the hunting side a map upon which certain directions are given. Such a game is played in much the same way as boys play "the hare and the hounds." When found, the hunted and hunters race back to the original starting point.

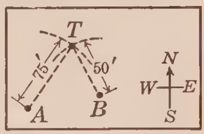
It is also a practical matter as well as an interesting project to discuss with the pupils how a person, by making use of the geometry of position, might avoid getting hopelessly lost in the woods.

Locating a Point in a Plane. Having now made some interesting discoveries for himself, the pupil is ready to consider more mathematically the question of locating a point in a plane. This can always be done if we know that the point is on each of two known lines which intersect, for it must then be at their point of intersection. For example, suppose that a farmer wishes to cover and bury a well in a rectangular field at a point T which is 50 ft. from the northern edge of the field and 75 ft. from the western edge, as shown in this figure. It is then a simple matter to relocate the point T at any time, for he needs merely to stake out the dotted lines as shown in the figure.

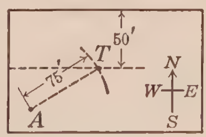


Instead of having two straight lines intersecting we may, however, have a circle intersect a straight line, or one circle intersect another. For example, the farmer referred to above might have located the well by means of the intersection of two circles, thus:

If, in the field, there had been two trees whose positions could be easily described, the man might have planned to cover the well which was located north of the trees at a point 75 ft. from the tree A and 50 ft. from the tree B. He could then relocate the point later by using a rope and taking a pointed stick to scratch on the ground intersecting arcs with radii of 75 ft. and 50 ft. respectively, as suggested by this figure.

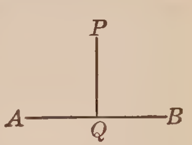


The farmer might also have located the well by means of the intersection of a straight line and a circle; thus, he might have remembered that it was 50 ft. from the northern edge of the field and 75 ft. from the tree A. He could then relocate the position of the point T as suggested by this figure.

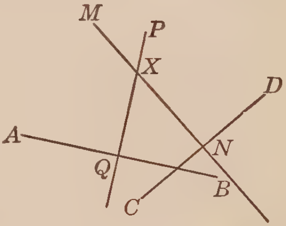


Points Equidistant from Two Points. After having shown how to locate a point by the intersection of two lines, it is desirable to show the pupil how other convenient lines can be found for this

purpose. He should, by a few simple questions, be led to see that if the line PQ is a perpendicular bisector of the line AB, all points in the plane that are equidistant from A and B lie on PQ or its prolongation.

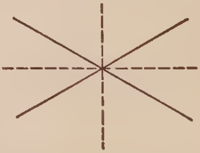


Therefore, if a point is equidistant from A and B it must, in this figure, lie on PQ; and if equidistant from C and D it must lie on MN. Hence if it is both equidistant from A and B, and from C and D, it must lie at X, the intersection of PQ and MN. If these lines do not intersect, there is no such point.



Points Equidistant from Two Lines. It is also very easy to bring a pupil to discover for himself the fact that points equidistant from two straight lines lie

1. On the bisectors of the angles formed by these lines, if the lines intersect, as here shown; or
2. On the line parallel to the given lines and midway between them if they are parallel.



General Law for Locating Points in a Plane. From the above discussion the pupil can readily see that *a point in a plane can be located if we know two lines upon each of which it lies, provided that these lines intersect.*

Locating a Point in Space. A point in space is located in somewhat the same way as we locate a point in a plane, except that we use intersecting planes instead of intersecting lines. The subject is not considered to any great extent in the junior high school, but a single illustration will be interesting to the pupils and will serve to show the general method.

If a point in the schoolroom is 8 ft. from the west wall, it lies in a plane that is parallel to that wall and is 8 ft. from it. If we also know that the point is 10 ft. from the north wall, it lies in a plane that is parallel to that wall and is 10 ft. from it. These two planes intersect in a line. If we also know that the point is

4 ft. from the ceiling, it lies in a plane that is parallel to the ceiling and is 4 ft. from it. This plane will cut the line just mentioned at the required point.

It is very easy to develop this with the class and to introduce other simple problems analogous to those already considered for locating a point in a plane.

QUESTIONS AND TOPICS FOR DISCUSSION

1. If we wish to teach anything besides arithmetic in the seventh and eighth grades, state which of the following would be most easily taught, giving your reasons:

   a. Algebra.
   b. Intuitive geometry.
   c. Numerical trigonometry.
   d. Demonstrative geometry.

2. Make a list of geometric forms which any citizen would naturally need to know.

3. Make a list of several words ordinarily used in the teaching of demonstrative geometry that might profitably be omitted in intuitive geometry.

4. What geometric constructions do most people need to know for use in daily life?

5. Make a list of the plane and solid figures which it is desirable for the educated citizen to know how to measure.

6. What does the educated citizen need to know in regard to symmetry and similarity?

7. Is there any value in the idea of a locus? If so, suggest one good way to introduce the topic.

8. Show how the work on the geometry of position may be correlated with the work in geography.

9. Read, if possible, Poe's "Gold Bug" to see what there is in the story that you might use in teaching the geometry of position.

10. How might a fisherman locate a good fishing spot by using four markers on the shore of a lake?

11. Make a list of the postulates and propositions that you would be willing to accept without proof in a course in intuitive geometry in the junior high school.

12. Discuss the method which you would use in defining the geometric forms used in a course in intuitive geometry.

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CHAPTER VI

THE TEACHING OF ALGEBRA

1. General Purpose in View

The Purpose of this Chapter. In Chapter III we considered the detailed objectives in the teaching of algebra. We shall now consider more fully the general purpose in teaching the subject, the essential topics, and such inherited material as has lost whatever value it may once have had and hence may be looked upon as nonessential or even as obsolete. This being done, we shall then consider some of the important questions in the psychology of algebra and their bearing upon practical teaching.

The Meaning of Algebra. The term "algebra" has had numerous meanings since it was first used in the ninth century, and the subject itself had various purposes even before it acquired its present name. It grew out of a desire for recreation, like chess and other games of skill; it developed thence into a branch of number theory; then it concentrated upon the equation; afterward it became a generalized arithmetic and was called by Sir Isaac Newton "arithmetica universalis"; and now, in elementary courses, it has become largely a study of formulas and their uses, equations being a means for carrying on the work. We have thus come to a more useful and definite purpose than was apparent a generation ago. Algebra is still, as it always must be, a science of numbers, — of general numbers represented by letters; it is the science by which we recognize universal rules which govern particular cases. For example, the formula for the square of the sum of two numbers, $a^2 + 2\,ab + b^2$, gives us a general rule covering all cases, whatever values we give to a and b.

Why Algebra should be Required. As to the purpose and need for teaching algebra, we cannot seek today to make a skilled

algebraist of every boy or girl, although skilled algebraists will be made. We cannot expect to have everyone solve two simultaneous quadratic equations of special types, although there will be a few who can do so. Nor can we expect to have all the pupils able to factor $ax^2 + bx + c$ (a useless accomplishment for most people), even though a considerable number will take pleasure in performing such a task and will thereby acquire some special skill which they will find useful in later work. The purpose of teaching algebra is found in none of these details; it consists in giving everyone a general idea of the meaning of algebra, together with a few definite and useful applications which everyone is likely to meet. If the subject is to be valuable, the learning should be a pleasure, and it may properly be expected that this pleasure will carry the pupil into such manipulations of algebraic expressions as will fix the habit of using algebra in the cases to which it can be applied.

Nothing has here been said as to the effect of algebra upon the formation of habits of accuracy and logical thinking, — habits that will unquestionably be transferred to nonalgebraic lines of work. It is not necessary for our purposes to do this. Whatever mental discipline is afforded by algebra can as well be developed by the useful side of the science as by the seemingly endless work with meaningless forms which so largely predominated in the older textbooks, and which, unfortunately, still finds a place in an occasional book of the present day and even in certain of the tests prepared by educators who have not carefully considered the objectives that should determine the details of teaching.

The American Point of View. One reason for the position above set forth is to be found in the peculiar conditions in education which face this country. It is the failure to understand these conditions that leads most European visitors who investigate our schools to say that our educational work is superficial. The charge is true. The work done in a French *lycée* or a German *Gymnasium* (two types of secondary schools in continental Europe) is much more thorough than the work done in our high schools. We sacrifice extreme thoroughness purposely, although regretfully, and in time Europe will have to do the same.

Bismarck objected to "an educated proletariat," and for the purpose that he then had in view (the German nation subservient to the crown) he was right. America, however, was established on different principles. For better or for worse — but we believe for better, and also believe that this will sometime be the feeling of all nations — we have demanded equality of privilege in education as we have demanded it in the courts and at the polls. We feel that every girl has the same right to an education as a boy, although the work need not be identical; that every woman has the same right to vote as a man; and that every child has the same right to go through a university as every other child, not only for its own sake but for that of the state as well. In other words, we believe in "an educated proletariat" as a safe foundation for our system of government. If we are to make the world safe for democracy, we must make democracy safe for the world, and the only formula we have devised for its accomplishment is the right kind of education. As to mathematics, the problem is to find what parts will best contribute to the end in view.

The decision to pin our faith upon education has necessarily brought a mixing of the intellectually backward and the more brilliant students in the same classes, and a consequent lowering of earlier standards. The difficulty is realized, however, and we have already made a good beginning by separating those whose attainments fit them for one type of school work from those who are better fitted for another. When this is accomplished, — and the progress at present is very encouraging, — we shall have better work in mathematics, better work in languages, better work in domestic science, better work in every line. As long as we insist upon a foundation that includes a fair degree of general information about the most important branches of human knowledge, of which mathematics is one, we can safely allow "individual differences" to determine to a large extent a student's future course.

Modern View as to the Best Course. In no mathematical subject has there been so great a change in recent years as in algebra, and in no period has there been a greater change in teaching

the elements of the subject than in this same period. In the ninth century, algebra changed from a subject dealing with puzzle problems solved by mere rule to being quite largely a subject in which the validity of rules was established; in the sixteenth century it changed from a subject concerned largely with rational numbers and the quadratic equation to one which dealt with irrationals and with higher equations; in the seventeenth century it changed from a subject with awkward symbols into one with a convenient symbolism and concerned with purely abstract formalism — work with polynomials which were rarely used and the solution of equations which had but little practical value. Great as these changes were, they were no greater than the change in purpose in American schools within recent years.

The basis of this change lies in the desire of the leaders in educational thought at the present time to segregate from the inherited mass of algebra those portions which the citizen will need either for practical use or for his general information, and to present these features to the pupil in a way that will secure and retain his interest.

Instead, therefore, of making the solution of abstract equations the central feature of algebra, as was once the case, or of making the formal juggling with algebraic expressions the main objective, as seemed to be the case in the first part of algebra a generation ago, the present plan is to emphasize the ordinary uses of elementary algebra and to introduce those features which add to the pupil's interest in the study of these uses. This has led to the placing of special emphasis on the formula (the most useful feature of algebra), directed numbers (at first only the positive and negative), graphs (a necessity at present in all lines of business and science), and the linear equation. Relatively little attention is now felt to be necessary with respect to the addition, subtraction, multiplication, and division of polynomials, to unusable cases of factoring, or to the manipulation of elaborate algebraic fractions. These topics kept the pupils busy and they had a certain interest for the very reason that they were mechanical operations requiring but

little thought; but their day is largely past, and were it not for the efforts of the ultraconservatives to keep them in state and city examinations, they would soon give place to the modern offering.

2. The Modern Course in Algebra

Essential Topics of Algebra. In order to see how we can best meet the needs of our future citizens, let us consider in a general way the topics about which everyone should have a fair amount of information. We shall then consider those in which it is not essential that every individual should be expert and those which may safely be discarded altogether.

The Formula. In the first place, if anyone uses algebra at all, the one part of the science that he will need the most is that relating to formulas. A boy who has a radio set and who takes enough interest in radio telephony to read about it in order to improve upon his apparatus even in some slight degree, will find that one of the first things he will meet in his reading will be an algebraic formula. It may be a simple one, but it will be algebra; and unless he knows the significance of the symbols and the way in which they should be used, his progress is stopped at once. Nor is this curiosity any longer confined to the boy; in a lesser degree, but still in a degree worth mentioning, the girl of today shows an interest in the "why" of this sort of apparatus. Although she may more reasonably be expected to meet formulas in a manual for nurses, in the modern study of dietaries, or in rules concerning the household budget than in problems of mechanics or in the economics of dairy farming, such cases are just as real as those met by the boy.

Thus, even outside the fields of commerce and of manufacturing, where formulas play an important part, both the boy and the girl come in touch with real needs for algebra that are of vital interest to them. The significance of the formula and its evaluation, the derivation of one formula from another, and the understanding that formulas are merely bits of shorthand which replace long and cumbersome rules, — all this is as much a part of the general information that the boy or girl should

have as nine tenths of what is taught in the average course in geography, general science, or literature.

How and Where to Begin the Study of Algebra. There are various possible ways of beginning the course in algebra. The old way was to consider the fundamental operations first, — a method which is now practically obsolete. We also hear it said, "Begin with equations because these are easy." Although this last assertion is true, it may not be honest. There is no doubt that children like puzzles and conundrums; that the pupil at this stage is largely controlled by "the wonder motive"; and that he delights in problems which start with such expressions as "I am thinking of a number." But though such an approach is easy, purposeless equations fail to lead to larger things in the way that formulas do. Like crossword puzzles they keep a pupil busy, but they simply lead into still more puzzles and conundrums. As a result this method of approach offers little chance for growth, and the work is usually seriously overdone.

The most hopeful way to begin the study of algebra is with the formula; for not only is this the most important part of algebra and the most purposeful approach to the subject, but it is doubtful if many persons will actually make use of any other part. It is doubtful whether most people ever really make any practical use of an equation except in connection with some kind of formula, and it is for this reason that we have come to make this feature, instead of the equation, the essential part of algebra, — the very heart and brain of the subject.

There are teachers who feel that the study of the formula is too difficult for pupils in the seventh or even the eighth grade. That depends entirely upon the manner in which the topic is approached. No one asserts that we should begin by talking about moments of inertia, acceleration, and the like to the pupils of a seventh grade, but it is very easy to introduce a few simple formulas without calling them by this name. We need merely say, "Today we are going to study a little interesting shorthand," and then proceed by using the initial letters of the important words to state a few familiar rules.

Not only is this introduction simple but it is particularly honest. There is a group of formulas that abundantly justify the teaching of algebra to everyone. In the seventh grade, under the head of "geometry of size," the teacher can develop certain mathematical rules and then teach the pupils how to translate them into the shorthand of simple formulas. This accomplished, the teacher should recognize that the first thing of importance is to find the value of the formula. The next thing needed is to know how to derive one formula from another. The pupil will need these two skills in almost any field, and the schools should clearly recognize this fact.

If we begin with a picture and develop such a formula as

$$A = \tfrac{1}{2}\,bh,$$

or as

$$A = \tfrac{1}{2}\,h(b + b'),$$

the work will be interesting, and the result will be usable. In this way we begin the study of algebra before we admit that we have reached the science itself. Such a study of even some of the simpler formulas given on page 176 puts the pupil into the way of functional thinking, — the essential feature of the course. As to the details of procedure, the teacher should study the model lesson on deriving one formula from another (page 302).

In our elementary textbooks we occasionally see the words "solve by algebra" given after some particular problem. There is much more to these words than at first appears, since they indicate that the pupil can just as completely (although not just as expeditiously) solve the given problem by arithmetic, and that he might, if left to himself, prefer to solve it in this way. As a matter of fact, one of the great advantages of algebra is that it furnishes us with a convenient machine for solving an arithmetic problem. This machine is the equation. A textbook in primary arithmetic that gives the expression

$$4 + * = 7,$$

and tells the pupil to write the correct number in place of the star, is giving an equation to be solved. We do not make the mathematics any more difficult when we write

$$4 + x = 7.$$

FORMULAS FOR CLASS USE

SUBJECT	FORMULA
Area of a rectangle,	$A = lw.$
Area of a parallelogram,	$A = bh.$
Area of a triangle,	$A = \frac{1}{2}\,bh.$
Area of a trapezoid,	$A = \frac{1}{2}\,h(b + b').$
Area of a square,	$A = s^2.$
Perimeter of a rectangle,	$p = 2(l + w).$
Perimeter of a square,	$p = 4\,s.$
Perimeter of a triangle,	$p = a + b + c.$
Perimeter of an equilateral triangle,	$p = 3\,s.$
Circumference of a circle,	$C = 2\,\pi r = \pi d.$
Area of a circle,	$A = \pi r^2 = \frac{1}{4}\,\pi d^2.$
Area of a regular polygon,	$A = \frac{1}{2}\,ap.$
Total area of a rectangular solid,	$T = 2\,lw + 2\,lh + 2\,hw.$
Total area of a cube,	$T = 6\,e^2.$
Lateral area of a circular cylinder,	$S = 2\,\pi rh.$
Total area of a cylinder,	$T = 2\,\pi rh + 2\,\pi r^2 = 2\,\pi r(h + r).$
Lateral area of a circular cone,	$S = \pi rl = \frac{1}{2}\,Cl.$
Total area of a circular cone,	$T = \pi rl + \pi r^2.$
Area of a sphere,	$S = 4\,\pi r^2 = \pi d^2.$
Volume of a rectangular solid,	$V = lwh.$
Volume of a cube,	$V = e^3.$
Volume of a circular cylinder,	$V = \pi r^2 h = Bh.$
Volume of a circular cone,	$V = \frac{1}{3}\,\pi r^2 h.$
Volume of a sphere,	$V = \frac{4}{3}\,\pi r^3.$
Area of a circular ring or annulus,	$A = \pi(r^2 - r'^2).$
The lever,	$w_1 d_1 = w_2 d_2.$
Distance, rate, time,	$d = rt.$
Thermometer,	$\begin{cases} F = \frac{9}{5}\,C + 32, \\ C = \frac{5}{9}(F - 32). \end{cases}$
Interest,	$i = prt.$
Amount at simple interest,	$A = p + prt = p(1 + rt).$
Amount at compound interest,	$A = p(1 + r)^n.$

Whether we shall call the first equation "arithmetic" and the second equation "algebra," or vice versa, is a matter of no consequence; it is a rather childish dispute over the meaning of words, since the process in both cases is identical. In other words, there is no sharp dividing line between algebra and arithmetic. By means of the symbols of algebra we are helped in solving the problems that formerly were part of arithmetic but which at present have been transferred to the more advanced field. There is no problem in elementary algebra that could not be solved by arithmetic if one had the time and the patience. The equation simply helps us to remember rules and to shorten the processes. Take, for example, such an unreal but quite familiar case as this:

What number is it that, increased by twice itself and decreased by 7, is equal to 23?

We may say that since a certain number was decreased by 7 to make 23, that number must be 30. We may then say that the number sought, increased by twice itself, is this same 30; that is, that three times the number sought is 30. We may then reason that if three times the number sought is 30, the number itself must be 10. We can now check our reasoning by observing that $10 + 2 \times 10 = 30$ and that $30 - 7 = 23$. We have thus solved the problem without any algebraic symbols.

If, instead of this long process, we accept the convenient shorthand of algebra, we may simply let x represent the number and write

$$x + 2x - 7 = 23;$$

whence

$$3x = 30,$$

and

$$x = 10.$$

Algebra, therefore, not only makes the process much clearer but saves us an amount of time that is worth considering.

So we say that one important thing in algebra which a pupil should know, for general information and for the purpose of learning how to solve problems and to save time in their solution, is the equation. The problem about numbers given above is, of course, nothing but a simple puzzle, introduced to show how a case that is thought to demand algebra can be solved

by arithmetic. Our modern algebras contain problems like the following:

In grinding wheat into flour there is a loss of 18% in weight. How much wheat, to the nearest 10 lb., must be used in making 1000 lb. of flour?

This too can be readily solved by arithmetic; but the solution by algebra is much simpler, and the same may be said of such problems as the following:

A mixture of 2 pt. of ammonia to 3 pt. of water is what per cent ammonia?

How much water must a nurse add to 1 qt. of a 20% solution of ammonia to reduce it to a 10% solution?

It should be stated frankly, however, that many of the illustrative problems in our elementary algebras are purposely fictitious, partly because certain extra-school examinations require it, and partly because the puzzle problem usually holds the interest of a pupil more than would a real problem from physics or from the field of industry or of commerce.

Inverse cases in percentage are usually not real cases, although the two given above are entirely so and have to be solved by trained nurses. One often has to know the answer before he can make the problem. The same is true of many motion problems, relatively few being found in practical situations. The problem about the planets in conjunction had to be solved at one time by someone, but for all but a professional astronomer it represents an interesting puzzle rather than a genuine application of algebra.

Types of Problems that still Persist. Let us consider three further types of applied problems that still persist in the textbooks:

1. *Coin problems.* These problems are useful because they represent a type which is needed in physics and which in itself is too hard for the pupil to solve.

2. *Digit problems.* These problems are interesting as puzzles. The realization of this fact is not shown enough in our teaching. It is better, however, to treat them by themselves instead of pretending that they illustrate the real applications of algebra. Their value lies in their interest and in the fact that they often

show the pupil how to solve another type of problem, which he may meet later.

3. *Pipe problems.* This problem is a relic from the time when fresh water was brought into a town by aqueducts and was conveyed to a large basin by one or more pipes. It is better than a problem about inertia, friction, and the like, which is usually stated in language that is hard for the pupil to comprehend.

The question then arises as to the types of problems that should be solved by algebra. The answer is that we cannot find many besides such types as those given above except in the field of formulas. It should always be recognized, however, that any problem that interests the pupil is valuable; and that if it fails to do so, it is practically valueless in his particular case.

Graphs. Besides the formula and the equation a third feature in algebra that boys and girls need to understand is the graph. They will meet with it everywhere in ordinary reading, usually as a bar graph or as a curve. What is needed in the junior high school, however, is chiefly the reading and interpretation of both statistical and mathematical graphs, not their construction.

The graph is seen in the advertising columns of our newspapers, in magazine articles on all sorts of economic and political questions, in various kinds of manuals used in the industries (including home economics), and even in summaries of athletic records and scores of games.

To be sure, much of the work connected with these types of graphs is mere drawing, and the mathematics is, as we have seen in Chapter IV, very largely arithmetic. It ceases to be so simple, however, when we come to consider the graph of the formula. Just as the formula is a rule represented by shorthand, so the graph is a formula represented by a picture. The formula makes the rule much more simple; the graph makes the formula much more vivid. We can draw a mathematical graph without knowing much mathematics, — the important connection with algebra is found in the picture which it fixes of the formula. Furthermore, a graph of a formula often gives an approximate numerical result at sight, whereas the formula

itself requires computation in each special case. An interesting case was suggested not long ago by an eminent physician. He had a formula which he had developed in connection with measurements of the resistance offered by the heart to an electric current. After laboriously working out a considerable part of a numerical table based upon the formula, he asked a teacher if there was not some way by which he could compute the rest of the table more expeditiously. As it happened, he was told that the formula was a linear function that could be expressed graphically by drawing a single straight line. In one minute the graph had saved him many hours of tedious labor. The case is not unusual. Many of the formulas that the modern farmer meets in books on agriculture can be represented by graphs which give numerical results to the same degree of accuracy as would be obtained with more difficulty by numerical computation.

Here, then, is another topic of algebra, the picturing of a formula, that has become prominent within a few years not only in school but in the domain of practical life as well. That is one of the best reasons for teaching it.

Directed Numbers. Next in importance after formulas, equations, and graphs come directed numbers. We lay out our cities and seat our pupils on an analytic-geometry plan. It would be considerably more convenient if we used $+120°$ or $-120°$ instead of 120° E. and 120° W. Such an illustration shows that a negative number is just as real as a positive one. To show this reality of such numbers requires merely the drawing of a picture. We therefore see that the fourth thing which everyone needs to know about algebra is that it extends the number system to include both the positive and the negative.

Primitive numbers were integers. It took thousands of years before the need for fractions developed, still longer before $\sqrt{2}$ was looked upon as a number, and many centuries thereafter before a number like -3 was needed in the affairs of daily life. Only a short time ago, as we reckon time in the progress of mankind, the statement that a thermometer indicated $-10°$ was meaningless; at present it is the common

method of showing that the mercury is 10° below zero. In the same way, we now speak of positive and negative latitude or longitude, of positive and negative forces, of negative scores in many kinds of games, or of a negative amount on a balance sheet as representing a negative profit (which is, of course, a loss).

While the negative number may be said to be mere arithmetic, the same could be asserted with respect to $\sqrt{3}$, the logarithm of a number, the sine of an angle, and the rule for compound interest. Most of these, however, are too advanced for the pupil who is studying elementary arithmetic, and so they conveniently find a place in algebra.

These four topics, therefore, — the formula, the equation, graphs, and directed numbers, — may safely be regarded as the minimum requirements necessary for an understanding of the meaning of algebra and as constituting the chief use of the subject on the part of the average well-informed citizen.

The Nonessentials of Algebra. It takes a long time for the schools to drop a custom that is once established. The algebras of twenty-five years ago were encumbered with a large amount of traditional material, much of which was retained because the subject was then taught for a purpose quite foreign to the one which we follow at the present time. The change in some of these features has been great; for example, we are well aware that the addition of the fractions

$$\frac{x^2 - 3x + 2}{x^4 + 6x^3 - 28x^2 - 46x + 35}$$

and

$$\frac{x^3 + 2x - 1}{x^4 - 7x^3 - 12x^2 + 13x + 5}$$

is never likely to be met by the elementary-algebra student in any field of applied mathematics, or even in advanced work in pure mathematics. It is a fictitious case, made up for the sole purpose of requiring a pupil to find the highest common factor of the denominators. Furthermore, we know perfectly well that the highest common factor was introduced for the sole purpose, in so far as elementary algebra goes, of finding the

lowest common denominator of two such fractions. We therefore have a kind of vicious circle: one useless thing is taught for the purpose of doing another useless thing, and the latter is taught for the purpose of using the first useless thing.

Our later school algebras recognize the futility of such work, with the result that the long-division form of finding the highest common factor has now been eliminated from all progressive texts. With this has gone the elimination of a large amount of work with difficult fractions which had no very cogent reason for being except that it kept the pupils busy.

Since the time when the first algebras were printed, there has appeared in the textbooks a great deal of work involving surd expressions like $2 + \sqrt{3}$. Up to 1900 the work kept increasing in volume and difficulty until it included cases like finding the square root of $7 + 4\sqrt{3}$ and the product of $3 + 2\sqrt{5} - \sqrt{7}$ and $5 + \sqrt{5} + 3\sqrt{7}$. Then the more advanced textbook writers began to see that if algebra was to be made a living subject, most of this material must give place to topics more suitable to the twentieth century. They recognized that for practical work $2 + \sqrt{3}$ was better replaced by $2 + 1.732$, or 3.732, a thing which was impossible when algebras were first printed, as the decimal fraction had not then been invented. With this came also the recognition of the fact that an equation like

$$x^2 - x + 3\sqrt{x(x-1)} + 2 = 0$$

is a fictitious affair, made up for the purpose of requiring a pupil to remember a certain trick solution. As a result of this awakening to the meaning and purpose of elementary algebra, the last quarter of a century has seen the banishment of such topics as the finding of the highest common factor by the long-division method, as mentioned above, a decided lessening of the work with fractions, and a recognition of the uselessness of much of the work with irrationals.

Fundamental Operations in Algebra. As stated previously, fundamental operations were formerly taught first in algebra and the work was greatly overdone. Today we introduce them after work on the formula, equations, graphs, and directed

numbers. Moreover, we have materially simplified the presentation of these topics and have reduced the amount. Even yet, however, the operations as given in a number of the current textbooks and tests are unduly long and tedious.

The early books on algebra gave little or no attention to such features as operations involving polynomials. The subject was looked upon as of little importance until sometime after books began to be printed, and it began to resemble our present work only in the seventeenth century. From that time until the close of the nineteenth century the work in the operations on polynomials continued to expand until it reached its climax. There then developed a strong reaction against the excessive use of these operations, and it was found how little there was in the subject that we could really justify. It was seen that we could advance no very valid reason for considering polynomials of more than three or four terms. Professor Thorndike summarized the opinions of leading teachers when he said, "In no subject of the high-school curriculum outside the mathematics courses is any need for the manipulation of complicated polynomials discovered, nor does any such need seem probable."

Factoring. About the year 1875 factoring was rarely found as a separate topic in any of our algebras, although of course this does not mean that people did not know the factors of expressions like $ax + bx - cx$ and $a^2 - b^2$. It then became the fashion to include it, and by the year 1900 one of the best textbooks gave nine different cases and four hundred and sixty-nine examples. The subject has some value, but nothing like the value that such figures indicate. At present only three cases are required (although they are usually elaborated into five), and our best books rarely give more than two hundred examples for drill, and our best textbook writers would probably be glad if the schools would allow them to reduce materially even this number. The subject is seldom needed except in connection with abstract problems so made up as to require it; for example, in reducing the fraction $\dfrac{x^2 + 4x - 21}{x^2 + 13x - 42}$

to its lowest terms or in solving (by a method that is not usable in most of the genuine applications of algebra) the equation

$$x^2 + 13x + 42 = 0.$$

One of the leading elementary algebras of about 1900 gave seventy-six examples in the factoring of $x^n \pm y^n$; one of our modern books very properly gives none. It is demanded in certain courses in the second year of algebra; but no pupil is likely to meet with it, for cases in which $n > 3$, in any practical way after leaving school.

It is desirable that pupils should know how to take out a common monomial factor or to factor the difference between two squares, but nearly everything beyond this has slight value at best. It is often difficult to get school authorities, or even practical teachers, to comprehend this fact. They frequently assert that the pupil should learn the method of factoring an expression like $12x^2 + 5x - 28$ in order to solve a quadratic equation by "the factoring method." It is interesting to observe, however, that these same teachers will then proceed to show pupils how to factor this very trinomial by solving the quadratic equation

$$12x^2 + 5x - 28 = 0,$$

thus completing an exceedingly vicious circle. As a matter of fact, the quadratic equations that we really find in physics are never of the artificial type that readily leads to solutions by factoring; they are almost invariably of the type

$$3.8x^2 - 4.2x + 2.8 = 0,$$

which can be solved far more easily by other methods than factoring.

The fact is that the chief practical use of factoring is to change certain expressions like $\pi r^2 - \pi r'^2$ into forms that can be more easily evaluated when, for example, $r = 4.7$ and $r' = 1.3$. It is quite safe to say that there is in our current algebras hardly a single case in the reduction of fractions or in the solution of equations by factoring that is not artificially constructed so that this method can be applied.

Fractions. What has been said about factoring leads to similar questions with respect to fractions. Where are they used?

What types do we really need? These questions are readily answered very briefly as follows:

1. We use fractions in the evaluation of practical formulas; but this can hardly be said to involve operations with algebraic forms. We also use them in reducing fractional equations to integral forms so as to render the solution more simple.

2. The types commonly used, both in formulas and in equations, involve only monomial or binomial denominators.

It will therefore be seen that much of the inherited work in fractions is artificial, at least so far as the needs of the pupil in elementary algebra and science are concerned. The argument that this inherited material throws light upon the work in arithmetic is fallacious. In the first place, the treatment of fractions required in most of our official courses of study is itself open to the criticism that much time is wasted upon the unpractical. An illustration of this fact is seen in such time-consuming work as

$$3\tfrac{3}{5} \div 2\tfrac{7}{8},$$

a case that never arises with such fractions in the lives of most people. In the second place, the only algebraic fraction that, through its manipulation, throws any light upon the arithmetic situation is the one with monomial terms. The pupil will find more light shed upon his arithmetic if he is thoroughly taught the use of this kind of fraction than he can find from a study of the complicated forms, the correlation in the former case being very close and in the latter one very remote.

It should be stated, also, that use of the complicated forms that are still seen is a recent matter. The early algebras gave none of it. The subject matter a generation ago was the creation of over-zealous textbook writers and of the makers of courses of study and of examinations set by outside authorities. Even our advanced mathematicians do not ordinarily need this inherited material; and if a case arises in which it appears, they can easily see how to treat it.

With the simplification of the fraction work in arithmetic there has disappeared the theory of greatest common divisor, least common multiple, and to a large extent of least common

denominator. This has been a great gain. We do not in practice need to add $\frac{3}{7}$ and $\frac{5}{9}$, and the practical cases in which addition and subtraction enter (say $\frac{3}{4}+\frac{7}{8}$) are those in which the proper denominator is readily seen. Similarly in algebra, for practical purposes the work in connection with the above theory has been greatly simplified, and for the better. The large amount of time heretofore given to algebraic fractions can profitably be reduced more than one half. Our better class of textbooks now recognize this fact and devote their attention to more profitable parts of the science.

Complex Fractions. In elementary work in science or in mathematics, not only in the junior high school but in the years which follow, the pupil will have no need for complex fractions beyond the simplest cases. If a pupil wishes to evaluate the formula

$$s=\frac{a(1-r^n)}{1-r}$$

for the case of $a=\frac{1}{3}$, $r=\frac{1}{2}$, and $n=5$, the formula involves a complex fraction. The chance of meeting with this case, however, is rather remote, and when he does meet with it the treatment is quite apparent, even if the pupil has never studied complex fractions as a special topic. In any case, all practical needs are met if we limit the work to such simple cases as the simplification of

$$\frac{\frac{a}{b}}{\frac{b}{c}},\qquad \frac{\frac{a}{b+c}}{\frac{b}{c+a}},\qquad \frac{\frac{a+b}{x-y}}{\frac{a-b}{x+y}};$$

and even these go beyond the demands that will be made upon the pupil's knowledge. The reduction of an engineering formula like

$$w=\frac{5\,a}{1+\frac{l^2}{600\,d^2}}$$

may well be reserved with a few exceptions for the technical training of the specialist.

Fractional Equations. That the fractional equation as it appears in our current teaching of algebra is very artificial is seen

from the fact that the applied problems in the topic are so few and so manifestly unreal. They are involved in the so-called "work problems," but not only are these problems rather artificial, but the amount of fraction work which they require is very slight. If the teacher will survey the work carefully he will find that the one great use of the fractional equation lies in the manipulation of formulas, and that here the demand is for only very simple cases. To solve the equation

$$s = \frac{ar^n - a}{r - 1}$$

for a represents about the limit of difficulty needed in practical elementary work. In order, however, to cover any demands that are likely to be made, we may extend the work a little farther, limiting it to those fractional equations in which the denominators are generally monomials or binomials.

Literal Equations. It will be observed that algebraic fractions, as distinguished from ordinary arithmetic forms, find their chief use in literal equations. When, for example, we wish to find a formula for b from the equation (formula)

$$A = \tfrac{1}{2}\,h(b + b'),$$

we need to solve a literal equation, and this becomes at once a fractional equation. It follows that the drill upon literal equations should be confined to those cases which assist in the manipulation of formulas and which involve no complications beyond the need imposed by them. This restricts the work to that which involves the kind of fractions mentioned in the preceding paragraph.

Simultaneous Linear Equations. Although the number of applications of simultaneous linear equations is great in the fields of science, commerce, and industry, those which are suited to the interests of the junior high school are few in number. This will be the more clearly seen by examining the problems given in the current textbooks. Because of their later use, however, it is desirable to give some attention to the method of solving them, this method being such as to be applicable in the field of genuine applications when that is reached.

These applications will not involve sets of equations like

$$2x - 7y = 11$$
$$3x + 4y = 29,$$

but rather those which have decimal coefficients like

$$3.05x - 4.32y = 9.87$$
$$0.72x + 0.39y = 3.46,$$

equations that would, if introduced at this time, center the attention upon tedious computation rather than upon the technique of solving.

We therefore see that while it is desirable that the subject of simultaneous equations should be taught because of their use in science, the fictitious nature of the applications suitable for the high-school pupil is coming to be recognized, and more attention is being given to a more careful selection. Thus, in one of our older books we have one hundred and eighty-nine applied problems on the subject, of which not a single one is of the slightest possible use in any practical line. Our modern books give a much smaller number and select well-known types, frankly admitting that the genuine applications belong to the technical fields of science and industry, which are beyond the pupil's present ability to understand. To omit the topic entirely is to deprive the student of something that he will need in his later work in science; to select problems from dietetics or physics is to render the subject too difficult; the only safe plan is to select problems that are as interesting as possible and to give the technique that will later be needed in the study of the sciences.

Quadratic Equations. Were it not for the fact that the quadratic equation is required to meet certain tests of the junior high school, it would not need to be taught at this time. Of course, such incomplete quadratics as $A = \pi r^2$ should have a place in connection with the study of formulas, but the solution of such a complete quadratic as $3x^2 + 7x - 26 = 0$ has no applications of importance in the junior high school. In the senior high school, in the study of intermediate algebra and in the preparation for the serious study of science, the subject has an undoubted place. Its treatment therefore in the

junior high school should be informational rather than practical, and increasingly less emphasis should be placed upon it.

Rearranging the Essentials. One great advantage of this elimination of the obsolete has been the opportunity that it offers of rearranging the retained material on a more rational basis. This is well illustrated by the tendency on the part of progressive writers to introduce the equation early in the course in connection with the study of the formula. This is when the need for it first arises, and where only very simple forms are required. When the student has advanced far enough to apply his knowledge to the solution of problems of greater difficulty, the equation is again considered, but on a higher level. Such a plan enables the student to review quickly the simpler treatment, thus refreshing his mind before the more advanced steps are taken, and at the same time improves upon the obsolete method by which a student studied this important topic only once and was then allowed to forget it.

Such changes cannot be successfully made overnight. We still retain a considerable amount of material that could be improved upon, but it is slight in comparison with what we had in the year 1900. We may therefore feel reasonably content with the notable advance that has already been made in the substitution of modern material for that which has long since served its purpose in our country.

The rearrangement of the essentials leads to the abandoning of the old method of discouraging the pupil by beginning with a large amount of formal addition, subtraction, multiplication, and division of polynomials involving only integers, then considering factors, and afterward reviewing all this work with fractional expressions. Instead of all this, the psychological arrangement is followed of taking in sequence simple formulas, equations, graphs, directed numbers, elementary operations, linear equations with one unknown, fractions, and so on to quadratic equations.

How to use a Textbook. In general, if the teacher has a good textbook, one that meets the modern demands, the best plan is to follow the sequence which it gives, enriching its discus-

sions and adding whatever of interest and value his experience can contribute. The textbook should be merely a guide and a helper, setting forth a reasonable sequence of topics and furnishing a sufficient amount of written work to prepare for any proper tests that may be given. It should act as a map for an exploring party of which the teacher is the leader, and should be consulted freely and often, not by the leader alone but also by all the members. It should never be felt that every pupil must solve every exercise; just as the head explorer assigns different duties to his associates, so the teacher should recognize the individual differences of his pupils. Some members of his class should be commended for solving ten exercises, where others should expect such recognition only after solving twice as many. Handicaps are as proper in algebra as in golf or tennis.

It is generally a good plan to develop briefly the advance lesson at the board, being sure that the probable difficulties are fully considered. Instead of this, it is sometimes as well to read, with the class, the explanation given in the textbook and to ask or answer such questions as arise. After this the problems for the advance lesson may be assigned and the work for the day may be taken up.

The Answer Book. In the affairs of life, whether they involve mere computations or elaborate problems, we have to check our work. Our grocery bills, our bank deposits, the change we receive, and our budgets and cash accounts, — all these have to be checked. If we have a complicated problem to solve in business life, we seek to solve it in two or three ways so as to check the conclusion and see that we make no mistake. Life furnishes no answer books.

In school, however, one of the legitimate checks is this very answer book which life declines to give us, and its use in many cases is commendable. In algebra, for example, results are often secured which are so complicated that the check is unduly burdensome. In such a case the pupil would use his time more profitably if he checked it by simply looking at the answer book. It is, therefore, the duty of the school to be certain that the pupil knows how to check every piece of work he does, and that he

actually does check those results in which the expenditure of time does not outweigh the advantages derived from the computation.

For the teacher the answer book is a great saver of time. Assuming that he knows his subject and can not only solve his problems but readily check his results, it would be a waste of his time and energies to do either, further than to be sure that he is master of the situation. In the same way, it is perfectly proper for the teacher to use freely a key, saving his energies for planning his lessons and for securing interesting material for his teaching.

As to the answer book, it is legitimate to allow pupils access to it if they wish to see the results, letting the consultation be perfectly "open and aboveboard." Since it usually takes longer to do this than to check a result independently, it is not likely to harm the good student, while it may be of real assistance to the poor one. Common sense should determine all such matters.

Value of Checking. On the whole, however, it is usually better for a pupil to solve one problem and check the result than to solve two and not check at all. The pupil then has a twofold gain: (1) he does a piece of work that is ordinarily quite as good an exercise as the original solution, and (2) he has the pleasure of being certain of his result and of his mastery of the entire situation. For this reason the exercise in which the subject of checking is introduced should be such as to lead to relatively easy checks. At this stage it is the technique of the check that is important, — the knowing how to accomplish the complete verification of an answer. In this way two major difficulties are not introduced at one time, and the checking habit develops gradually, beginning with simple cases and increasing the burden slowly.

Board Work. When a pupil is sent to the blackboard to work or to explain a problem the teacher should insist that he "chalk and talk." Such a practice not only saves time but helps to keep the rest of the class attentive. It also makes it possible for the pupils to follow what is being done and to get the most out of the recitation. This method of explanation is

often difficult, but it is an ideal that is well worth working for diligently. The pupils should be permitted to refer to notes in such work just as adults often do when speaking in public. It is rarely necessary or desirable to memorize the work; it is far better to encourage the pupil to talk naturally as if in friendly conversation.

When a problem or principle is being explained it is best to have only one pupil at the board at a time, and if the problem is a difficult one, one of the more able pupils should be allowed to assist, — which is better than being asked (much less commanded) to do so. For the teacher to select an intellectually backward pupil for an explanation or a discussion, — one who is sure to fail, — is manifestly poor policy, resulting as it does in discouragement and absolute waste of time.

If a new topic is being developed, teachers often like to send a part or all of the class to the blackboard to see if the pupils have generally grasped the idea. This is especially true in drill work. Much time and energy is often wasted, however, in passing to and from the board. If all the pupils work at the board on the same problems, some are so much faster than the others that class unity is often lost, bad habits of copying other pupils' work are begun, and general confusion results. Moreover the classes in many of our schools today are so large that there is often room for only a few pupils to work at the board, and the managing of the entire class then becomes a serious problem. Added to these difficulties is that the board work is also dusty and unhygienic and that practically all the desirable results can be secured by having the pupils do their work on paper. In view of the new types of work books and drill exercises, it would seem that the use of blackboards is likely to be reserved largely for the teacher or for the single pupil who is called upon to assist by some explanation of his own.

3. The Psychology of Algebra

Why Pupils Fail in Mathematics. It is impossible for us to justify the large percentage of failures in mathematics reported by so many of our schools. Whenever the number of failures in a

sufficiently large group exceeds 10%, the reason should be found. It is true that we must not expect universal success in mathematics any more than we expect it in life, but it should be possible to avoid mental bankruptcy in the former and financial bankruptcy in the latter. In each case it should be possible to search out the cause and to apply the necessary remedy.

In educational parlance, the fact of individual differences explains why some people succeed better than others. The standards set by the schools, however, should not be such as to retard the progress of the large number of pupils who constitute what is technically known as the modal group. Indeed, even the mentally inferior group should be encouraged to advance as far and as rapidly as their powers permit, just as should also the intellectual leaders, even though the progress be at a different pace and the ground covered be materially less.

It is true that the number of pupils attending the junior high school has greatly increased during the past few years, both actually and relatively, and their native capacities, life experiences, and purposes of study are much more varied. The controlling cause of the difficulty, however, is not here; it is to be found rather in our failure to make a careful analysis of our objectives and to fit these objectives successfully to the various types which our diagnostic tests determine.

The problem is by no means impossible of solution. The success in improving the quality of instruction in arithmetic proves that. If we can better the work in that subject, we can do so in algebra and, indeed, in all the other branches of mathematics. We shall therefore consider briefly a few of the major difficulties which pupils encounter in algebra and shall then suggest a few of the more important remedial measures to be taken.

Difficulties in Reading Algebra. One of the chief causes of difficulty in the study of algebra is found in the failure of pupils to read understandingly, and one of the chief causes of this failure lies in the lack of a sufficient vocabulary. It is due to this fact that careful studies have been made in recent years of the minimum word lists to be expected of pupils in various branches of study, algebra included. Our best modern writers

have taken the results into consideration and have sought to treat of algebra in simpler language than was the case when only the most gifted attended the high schools.

The teacher of this branch as well as of others should always remember that the greatest works are clothed in simple language. The greatest teachers have spoken to the world in words that children can comprehend. The small mind seeks to use difficult words; the large mind seeks to use simple ones.

This study of the proper vocabulary leads us into many interesting investigations. For example, it is natural for an adult to ask what number is seven "more" than nine, a problem leading to addition. It is equally natural for him to ask how much "more" nine is than seven, a problem leading to subtraction. To the child, however, who hears the key words "more," "nine," "seven," it is often very confusing to know whether he is to add or subtract; indeed, he may quite possibly think of multiplying to find a number "more" large, or of dividing to find how many "more" one number is than the other. On this account the beginner has a right to expect such simplicity of language as will leave in his mind no doubt of the meaning involved.

A second cause of difficulty is found in the technical language of mathematics. It is for this reason that we have abandoned the use of the term "affected quadratic," the first word of which conveyed no meaning to teacher or child, and the second of which meant little to the pupil unless his teacher foresaw his probable difficulty and removed it. It is, therefore, desirable to simplify technical language as much as we reasonably can, occasionally putting in place of the difficult word itself the meaning expressed in the language of childhood.

Related to the difficulty arising from the technical language of mathematics is that which is due to symbolism. This does not ordinarily cause as much trouble in algebra as in geometry, where the desire to introduce unnecessary symbols seems to be almost a mania with certain teachers. It is, however, serious enough in the former subject, some instructors resorting to symbols that have no sanction in the mathematical world

and no worthy reason for being in their own classes. The subconscious purpose is probably to appear original; the very conscious result is to add to the pupil's burdens.

Finally, the beginner in algebra may have trouble because he fails to use all the necessary data in a problem. Even though most textbooks state precisely the needed data and no more, pupils will frequently fail to use all of them, the result being necessarily a failure to reach a solution. If we should state problems with insufficient data, or with some that is irrelevant, as some writers have attempted, the situation would thereby become even worse. The only remedy for the difficulty seems to lie in giving a great deal of practice in problem reading after the manner of some of our best silent-reading devices.

Establishing Unnecessary Habits. In education in general and in algebra in particular it is a bad practice to establish two or more habits when one is sufficient. For example, it is not desirable to teach a beginner two methods of algebraic subtraction, four methods of factoring $15x^2 - 14x - 49$, or four methods of solving a quadratic equation. Later in the course, various methods of doing a specified thing may be discussed, but for a beginner it is desirable to develop one plan for each case and adhere to that until its use has become automatic. What might be called the development of uncertain attack merely wastes time and confuses the pupil.

Avoiding Incorrect Transfer. The pupil beginning his work in algebra brings with him certain ideas that are correct in arithmetic but with which he is liable to fall into error when he applies them to algebra. For example, in arithmetic he has acquired the habit of proceeding from left to right in a case like $2 + 3 + 9 - 4$, a rather useless but common problem. He carries the same principle to the case of $a + b \times c - d$, although the usage in algebra leads us to perform the multiplication first. As a result, he tends to multiply $a + b$ by c, and then to subtract d.

Similarly, every teacher has found that pupils often tend to cancel the a's in a case like the following:

$$\frac{a+b}{a} = \frac{\cancel{a}+b}{\cancel{a}} = b.$$

To avoid such incorrect transfer it is essential to provide abundant practice in attaining the correct objective in every typical case.

Failure to prepare for Future Needs. While it is a good principle to prepare for a particular skill just before you reach it, this is not always the best way. In addition, for example, we prepare for several things, and it is not feasible to place addition immediately before each. By way of illustration, in solving equations we need to add and to subtract horizontally, but it is not desirable to hold this work until this particular need is felt; it is much simpler and more natural to prepare for this future need while we are on the special subject of addition. With only a little practice the pupil can add one-figure numbers horizontally as easily as vertically, and this should form a part of the work in addition.

Failure to prepare for Correct Induction. Every teacher of algebra has probably been disturbed because some of the pupils fail to generalize correctly from particular cases; that is, they fail to make a correct induction. If a pupil is ever to learn to make his own generalizations when alone, he must have plenty of practice in discovering probable general rules from special cases and in stating these rules accurately. This does not prove the rule; it is merely a method of discovering the probable. It is then that the mathematical mind undertakes its particular task, — that of proving that the probable is really a certainty.

Obviously not all children are equally gifted in any one thing, — for example, in this same power of generalizing. The teacher's tendency in the case of this type of failure as in most others is to neglect the mentally inferior pupil and to give only the mentally superior ones the opportunity of expression. Time is valuable, and the teacher will often give the generalization himself, without waiting for anyone to show his own originality. Such a procedure, however, is not justified. In the long run more time is wasted by undue haste than by concentrated practice on desirable habits, such as reducing the number of steps to a minimum in arriving at some particular generalization.

Difficulty with Inverse Cases. There exist in mathematical problems, and particularly in arithmetic, what are known as inverse cases. The name is meaningless except as we say that one operation is arbitrarily called direct and the other is then its inverse. For example, if we say that $2 \times 3 = x$ represents a direct case, then $2 \times x = 6$ may be said to be its inverse; but if we say that $2 \times x = 6$ represents a direct case, then $2 \times 3 = x$ is its inverse.

The term is frequently used in connection with percentage, and

$$5\% \text{ of } \$100 = x$$

is spoken of as the direct case. This being so,

$$5\% \text{ of } x = \$5$$

and

$$x\% \text{ of } \$100 = \$5$$

are inverses of the first. Such cases in arithmetic occasionally present some difficulties to the child. He knows that he is to divide but does not know which is the divisor. As soon as he has the algebraic symbolism to assist him, however, the difficulties are negligible; he has merely a very easy equation to solve. Other similar cases, all involving essentially the simple equation

$$ax = b,$$

are treated in the same manner. Manifestly, however, algebra is not essential in the treatment of cases of this kind.

Failure to Analyze Objectives. One of the most frequent failures in the teaching of algebra is due to the tendency to see an objective as a single entity, when as a matter of fact it may seem to the pupil very complex. The solution of a mathematical problem is rarely an exercise of a single skill; it almost invariably requires not merely several skills but the ability to coordinate them. To take an example from arithmetic, a child may have perfect mastery of the six skills involved in

$$4 \times 5,\ 4 \times 8,\ 4 \times 2,\ 6 \times 5,\ 6 \times 8,\ 6 \times 2,$$

besides those involved in simple additions, and still be unable to multiply 582 by 46, an operation which requires precisely these elements. He fails in the coordinating of all this work, and the teacher fails in neglecting to drill upon it.

Similarly in algebra, we not only need to see clearly the great objectives but we need to analyze each into unit skills and then to be able to coordinate these on a working basis. The teacher should see that the pupils be given a chance to make every possible error that can occur in attaining the objective under consideration. In this way his deficiencies can be discovered and the remedy can be applied. This remedy consists in sufficient practice, particularly in the region of defects, to insure satisfactory mastery of the process.

As an illustration of the failure to master certain of the fundamental skills in algebra a recent study[1] shows that 74.4% of a group of over 600 ninth-grade pupils could simplify the expression $7x + (-3x)$, and that 70.9% could perform the following subtraction:

$$\begin{array}{r} +8a - 12b - 156 \\ \underline{-5a + 7b - 75} \end{array}$$

In other words, the former was nearly as difficult as the latter. The same study shows that for the same group of pupils it was considerably more difficult to simplify $4x - 6y - 5x - 3y$ than to perform the subtraction

$$\begin{array}{r} -ab + a^2 - 3b^2 \\ \underline{+ab - a^2 + 2b^2} \end{array}$$

even though the latter involved the subtraction of $+2b^2$ from $-3b^2$, which should be as difficult as anything in the former. In the first case 29.3% of the answers were correct, whereas in the second case the score was 65.3%.

There are doubtless various possible explanations of the facts cited above. That there should be such a difference in the scores shows how marked had been the failure to analyze objectives and to drill adequately upon the constituent skills involved.

Method of Analyzing Objectives. The discussion of the preceding topic was destructive rather than constructive. The purpose was to show the faults of the failure to analyze. It is now proposed to make the criticism constructive and to show

[1] W. D. Reeve, A Diagnostic Study of the Teaching Problems in High School Mathematics, p. 107.

how it is possible to analyze objectives in such a way as to render more probable the development of all the skills involved in an operation. Because of the difficulty attending algebraic subtraction, and of the careful studies that have been made of the subject, we shall consider the case of subtracting one integral monomial from another similar one. In this operation there are four typical situations to be considered, as follows:

1. *Subtraction of simple directed numbers.* Under this head thirty-three major types are considered. In the tables, m is the absolute numerical value of the minuend; s, of the subtrahend; and d, of the difference. The symbol (+) means a positive number with the sign expressed, like $+5$; [+] means a positive number with the sign not expressed, like 5; and similarly for (−) and [−]. Cases in which $d=1$ are omitted in Table I, as are also the special cases of $m=s=1$ and $m=s=0$, although for completeness they should be considered.

2. *Subtraction of one monomial from another, a single letter, no exponents expressed.* Under this head sixty-eight major types are considered. The remarks under the preceding case apply to this one, except that the letters there given relate to the coefficients of the expressions.

3. *Subtraction of one monomial from another, two or more letters, no exponents expressed.* Under this head the sixty-eight major types of the preceding paragraph are included. With some pupils the transfer will readily be made from a case like $8a-4a$ to $8ab-4ab$, so that this paragraph does not usually lead to sixty-eight new types. For others, however, the transfer is not immediately evident, and in this case considerable drill may be required.

4. *Subtraction of one monomial from another, a single letter, an expressed exponent greater than unity.* As in the preceding case, sixty-eight major types need consideration, and as in that case there will be some transfer from Nos. 1 and 2, but this will not be so apparent as in the case of No. 3. The reason for this last statement is found in the fact that pupils will often subtract the exponents, just as they often add them in a case like adding $8a^2$ and $4a^2$.

TABLE I

SIGNS	$m > s$	$m < s$	$m = s$	$m = 0$	$s = 0$
+ *expressed*					
(+) − (+)	+ 8 + 6	+ 4 + 9	+ 5 + 5	0 + 3	+ 3 0
(+) − (−)	+ 8 − 6	+ 4 − 9	+ 5 − 5		
(−) − (+)	− 8 + 6	− 4 + 9	− 5 + 5		
+ *omitted*					
[+] − [+]	8 6	4 9	5 5	0 3	3 0
[+] − (−)	8 − 6	4 − 9	5 − 5		
(−) − [+]	− 8 6	− 4 9	− 5 5		
(−) − (−)	− 8 − 6	− 4 − 9	− 5 − 5		
+ *expressed once*					
(+) − [+]	+ 8 6	+ 4 9	+ 5 5		
[+] − (+)	8 + 6	4 + 9	5 + 5		
− *expressed once*					
0 − (−), (−) − 0				0 − 3	− 3 0

The Experimental Work. In an experiment recently made in Grade VIII, after the types of difficulty in the four situations described above had been analyzed, it was decided to teach the subtraction of one simple directed number from another, basing the work on the analysis made in the first case. Two days were spent in teaching this topic and all the various types of difficulties were discussed in class. On the third day a test, constructed

TABLE II

SIGNS	$m>s$	$m<s$	$m=s$	$m=1$	$s=1$	$m=s=1$	$d=+1$	$d=-1$	$m=0$	$s=0$
+ *expressed*										
(+) − (+)	$+8a$ $+6a$	$+4a$ $+9a$	$+5a$ $+5a$	$+a$ $+7a$	$+3a$ $+a$	$+a$ $+a$	$+5a$ $+4a$	$+4a$ $+5a$	0 $+2a$	$+2a$ 0
(+) − (−)	$+8a$ $-6a$	$+4a$ $-9a$	$+5a$ $-5a$	$+a$ $-7a$	$+3a$ $-a$	$+a$ $-a$				
(−) − (+)	$-8a$ $+6a$	$-4a$ $+9a$	$-5a$ $+5a$	$-a$ $+7a$	$-3a$ $+a$	$-a$ $+a$				
+ *omitted*										
[+] − [+]	$8a$ $6a$	$4a$ $9a$	$5a$ $5a$	a $7a$	$3a$ a	a a	$5a$ $4a$	$4a$ $5a$		
[+] − (−)	$8a$ $-6a$	$4a$ $-9a$	$5a$ $-5a$	a $-7a$	$3a$ $-a$	a $-a$				
(−) − [+]	$-8a$ $6a$	$-4a$ $9a$	$-5a$ $5a$	$-a$ $7a$	$-3a$ a	$-a$ a				
(−) − (−)	$-8a$ $-6a$	$-4a$ $-9a$	$-5a$ $-5a$	$-a$ $-7a$	$-3a$ $-a$	$-a$ $-a$				
+ *expressed once*										
(+) − [+]	$+8a$ $6a$	$+4a$ $9a$	$+5a$ $5a$	$+a$ $7a$	$+3a$ a	$+a$ a	$+5a$ $4a$	$+4a$ $5a$	0 $2a$	
[+] − (+)	$8a$ $+6a$	$4a$ $+9a$	$5a$ $+5a$	a $+7a$	$3a$ $+a$	a $+a$	$5a$ $+4a$	$4a$ $+5a$		$2a$ 0
− *expressed once*										
0 − (−), (−) − 0									0 $-3a$	$-3a$ 0

so as to include all of these types of difficulty, was given to a class of twenty-four children. Neglecting a few chance mistakes, which will always be present, we may say that the pupils had mastered the topic, at least for the time being, in a satisfactory manner.

It was then decided to find out how well the pupils could master the various types of difficulty shown in the second, third, and fourth situations described above, without any further teaching of the idea of subtraction. This was done in order to ascertain to what extent transfer could be trusted to produce satisfactory results. No reference was made in class to these cases, and tests were constructed so as to give the pupils a chance to master each type of difficulty. In so far as the particular class tested is concerned, it is safe to say that we cannot rely upon transfer for all pupils. The superior ones will usually make the transfer, but the others will often have to receive special attention from the teacher.

Some of the superior pupils who had been taught how to subtract $+12$ from -8 were successful in subtracting $+12x^2$ from $-8x^2$, while in certain other cases the pupils subtracted one exponent (2) from the other, became confused, and did not know how to proceed further. A few thought that inasmuch as 2 from 2 is zero the entire result should be zero. These are merely examples of many similar mistakes that were made in a relatively small group of twenty-four pupils. It is evident that in a larger group more difficulties would appear.

The results show that, after division has been studied, all types under the fourth case should be given special consideration in order to insure a better understanding of the process.

Summary of the Experimental Results. While a similar experiment involving a larger number of pupils should be carried out and the method extended to other objectives than those in subtraction, the results described in part above lead to the following conclusions:

1. In the subtraction of similar integral monomials there are apparently 101 typical difficulties (and probably 169 or more for many pupils) which are different enough to be considered separately. If we include such types as $0 - 0$ and $\pm 1 - (\pm 1)$ we shall increase the number to over 170. It is possible that a more complete experimental analysis will show some of the skills herein discussed to be mere duplications, but the facts in the case should be determined by actual trial in the classroom.

2. In spite of the difficulty for certain pupils, it is evident that many teachers have not considered the attainment of the subtraction objective a difficult one. Mathematically it seems simple enough; psychologically this is not the case. This is shown by a recent attempt of a certain teacher to secure satisfactory mastery in subtraction by developing with his class only three of the possible thirty-three cases shown in Table I on page 200. The teacher in question used the following examples as a basis:

$$\begin{array}{r}+8\\ \underline{+5}\end{array} \qquad \begin{array}{r}+8\\ \underline{-5}\end{array} \qquad \begin{array}{r}-8\\ \underline{-5}\end{array}$$

He then announced the conventional rule for algebraic subtraction, evidently without realizing that there might be trouble later for certain pupils if they were asked to subtract $+6$ from 0, or $-4x^2$ from 0, or even $+6x^2$ from $-2x^2$. In fact, a diagnostic test would no doubt reveal many difficulties in the path of his pupils. Such tests will be discussed in detail in Chapter XI.

The teacher referred to had not kept in mind (if he ever knew it at all) the proper method of an inductive approach to the rule and the need for analyzing an objective into its various components.

Recommendations. We need first to arrange the subject matter of algebra possessing known and varying amounts of necessary practice in terms of the unit skills involved. After this is done, we need to study the achievements of our pupils in relation to the practice they have received in trying to attain mastery.

If they need only a little practice to achieve satisfactory mastery (and we should determine within reasonable limits what this is), we should not keep them practicing on certain skills beyond the stage of diminishing returns.

Objectives needing Analysis. It is possible to show how to teach many traditional topics better than we do, but certain of these things are not worth the teaching. It is therefore necessary first to select our objectives with care and then to analyze completely only those that are valid. This being done, we should, as far as we are able, make an analysis showing how

much each unit skill or group of such skills needs to be made the subject of drill. The next step is to arrange the subject matter so as to secure the best results.

The procedure is relatively so new that any more detailed statement is not worth attempting. A complete analysis of all the objectives has not been made, and it is possible that it may not be worth making, or at any rate worth taking seriously further than to profit as much as we may from a study of the results.

Failures Due to Forgetfulness. Much trouble and a certain percentage of failures are doubtless caused through forgetfulness of what was learned in arithmetic or in previous topics in algebra. It certainly cannot always be maintained that the preliminary work was poorly done; indeed, such an assertion with respect to particular instances of failure is rarely valid. Because pupils in Grade IX cannot add fractions, as is often the case, it will not do to say that they never learned the process well. The difficulty lies not in the original teaching but in the failure to continue the drill upon the subject at frequently recurring intervals. Indeed, a test at the beginning of the year, followed by remedial drill, would have removed the defect. If a teacher of algebra lays blame upon the teacher of arithmetic in the preceding year, he should do so with the assurance that some college instructor will blame him in a similar way.

4. Criteria in Choosing a Textbook

Textbooks in General. We have reserved until this time the question of the choice of textbooks, partly because it is now possible to view the whole situation more intelligently and sympathetically, and partly because it is in the subject of algebra that the pupil begins to appreciate a well-ordered sequence of topics. What we shall have to say upon the criteria for judging a textbook in algebra, however, will apply with slight and evident modifications to a geometry, to books upon general mathematics, and to those texts which are prepared solely for the three years of the junior high school. We shall, therefore, as-

sume that the teacher will read this section with a view to its application to textbooks in general.

Points to be Observed in Choosing an Algebra. One of the most important problems that the teacher of algebra must solve is that of choosing a usable textbook, — a far more important problem than many teachers realize. No one would think of asserting that one person is just as good as another to select as a friend, and the comparison is not at all far-fetched when we say that it is equally true that not every textbook is just as good as any other.

The choice is not one that interests the teacher alone, for upon it rests not only the welfare of the pupils but, so far as they are concerned, the very existence of algebra. The list of points to be observed in the choice of a textbook, as set forth below, will assist many teachers in making a wise choice. It relates not to the merits of any special book but to a growing class of books which are modern in spirit and worthy in their construction. It is the result of the combined judgments of a large number of experienced teachers and may therefore be looked upon as at least fairly authoritative and representative.

General Considerations. There are certain general considerations that enter into the choice of any textbook. These are

1. *The purpose of the book.* Under the head of purpose there arise the following questions:

a. Are the author's aim and particular point of view definitely set forth in the preface?

b. If so, does the aim conform to the judgment of scholarly writers of the present time, — those who lead in training teachers to impart a knowledge of sound mathematics in the most approved manner?

c. Does the book reveal clearly the purpose of the particular subject treated, in this case algebra, thus giving to the pupil an appreciation of its importance in the lives of people in general?

d. Does the table of contents show a well-ordered sequence of topics, psychologically planned and so arranged as to show the significance of the work as a whole?

2. *The realization of this purpose.* To ascertain whether or not the author has realized his avowed purpose, it is well to ask one's self these questions:

a. Is the author's aim, as set forth, realizable in practical teaching? An aim may be too high to allow of hitting the mark, or it may be too low, or it may be wholly lacking in accuracy.

b. Does the book reveal to the pupil the nature of the subject, — say algebra, arithmetic, or geometry? Does it make clear its importance, and its interesting features?

c. Is the book so constructed that it reveals the purpose of the subject in such a way as to give some assurance that the pupil will realize this purpose and will be inspired thereby to put forth his best efforts in his study of the subject?

3. *Subject matter.* It is exceedingly important that the nature of the subject matter be carefully scrutinized. This does not mean the mere counting of exercises or of problems, — a very crude way of reaching any conclusion whatsoever. It lies rather in considering the bases of selection of the material, and this is revealed by such questions as the following:

a. Are the recommendations of recent national studies and reports followed with respect to the vital topics to be considered? For example, with respect to algebra the report of the National Committee on the Reorganization of High School Subjects recommends the formula, equations, graphs, directed numbers, and an introduction to trigonometry as the topics of greatest importance.

b. If an algebra or a geometry is being considered, does the book emphasize these topics or does it contain merely the inherited material relating to operations or propositions that are no longer looked upon as essential?

c. Is the material related to the real needs of people? For example, are the formulas of algebra such as the pupil will need in reading books on elementary science, on the making of a radio set, or on ordinary business practice? In other words, is the material socialized as much as is reasonably possible?

d. Does the material give evidence of having been tested in the classroom, either by the author or under his direction?

e. Has the material been chosen after a careful consideration of the relative value of the several topics? For example, in the case of algebra has the author unduly emphasized the four operations with abstract expressions, — a line of work of very little value; or has he given relatively more attention to such an important topic as the interpretation of the meaning of graphs?

f. Are the main topics made interesting and full of meaning by the inclusion of helpful details? For example, has the chapter on numerical trigonometry been so presented as a part of algebra that the pupil sees it develop from the ancient shadow reckoning, and gains some slight idea of the way in which great distances are measured?

g. Using modern accepted objectives in the teaching of algebra (pages 67–84) as a guide, do you find that all obsolete and useless material has been omitted so far as present conditions allow? Such unimportant material includes traditional work that is not used in modern science and industry or in some part of advanced mathematics that is itself valuable.

h. Is the material so selected as to provide for a sufficient number of exercises to meet the needs of all types of pupils? In other words, are the individual differences of pupils and of groups sufficiently provided for through an abundance of material of a kind that we can fully justify?

i. In the selection of material is a proper balance maintained between the concrete and the abstract? For example, a book that has five hundred examples in the relatively unimportant subject of factoring, without a single genuine use of the topic, is wanting in balance. Similarly, a book that has four or five thousand abstract examples, with substantially no genuine applications, is hopeless. Mechanical efficiency has very slight value unless someone is going to use it somewhere and sometime.

j. Are the topics limited in the selection of material, so as to present the most important ones, treating these intensively?

4. *Arrangement of material.* After the appropriate material has been selected, there arises the necessity for arranging it to

the best advantage for teaching. The following questions should therefore be considered:

a. Is the book made up of a series of unrelated chapters, seemingly without any connection; or is there a unifying principle that is increasingly apparent as the pupil proceeds? For example, referring again to the subject of algebra, is the dependence of one quantity upon another (the idea of functional relationship) made sufficiently prominent in the study of the formula, the graph, the equation, ratio, proportion, and variation?

b. Are the topics arranged according to a psychological sequence rather than merely a logical one?

c. Does the book follow the modern arrangement of topics, so that the most interesting and most frequently used ones come first and the more abstract and less frequently used ones come later?

d. Is the material cumulative in arrangement, each part in general being used in what follows? This need not interfere with arranging certain minor details with a view to their omission without destroying the main sequence.

e. Is the material so arranged that each type of difficulty is introduced on the principle of "one at a time"?

f. Is each new topic followed immediately by carefully graded exercises, sufficient in number for the purpose of fixing the principle, but not so numerous as to become mere dull routine material for keeping pupils busy?

g. Are the necessary definitions introduced when and only when the need for them is felt?

5. *Method of treatment.* Having considered the material and its arrangement, the method of treatment demands such attention as is suggested by the following questions:

a. Is provision made for the gradual learning of terms and processes, so that no avoidable congestion of difficulties shall occur at any one time?

b. In algebra, is the equation made subsidiary in importance to the formula? The pupil should see that the chief use of the equation is to enhance his ability to handle some kind of formula

that he will meet with in science, in commercial mathematics, in measurement, or in advanced mathematics.

c. Does the book provide for a simpler treatment of certain traditional topics than was formerly the case? As to algebra, this question refers to such matters as the use of parentheses, simultaneous equations, the quadratic equation, factoring, fractions, and especially the four operations.

d. Is the checking of all work emphasized and thoroughly explained?

e. Is provision made, through correlation, generalization, and application, for transfer of abilities to other subjects and from topic to topic?

f. Do the explanations foster the ability to read the book understandingly?

g. Is the language of the book so simple as to be understood easily by the reader? Are the explanations stated in words and sentences that are readily comprehended?

h. Is the significance of the work rather than mechanical skill in unimportant operations emphasized?

i. Is the pupil led to see a worthy reason for each new topic? In pedagogical language, is the material well motivated?

Special Features. Besides the general considerations that enter into the evaluation of a textbook, there are certain special features that demand attention. These are as follows:

1. *Applied problems.* Whether in algebra, arithmetic, trigonometry, or intuitive or demonstrative geometry, the nature of the applications demands the consideration of the following questions:

a. Are the applied problems as useful as possible, considering the pupil's limited knowledge of science, industry, and commerce?

b. Are these problems sufficiently numerous to emphasize the practical side of the subject under consideration, and is their importance in this regard duly emphasized?

c. Does the book recognize that problems may be genuine, in that they may be necessary in astronomy or in chemistry, and yet not seem real to the pupil at his present stage of advancement?

d. Does the book also recognize that a problem, say of the puzzle variety, may be very interesting to the pupil, leading him to acquire the technique of solution, and still be not at all genuine and possibly be uninteresting to the teacher?

e. Are the problems clearly stated, so that there shall be no linguistic difficulties in addition to those of a mathematical nature?

f. Are the problems selected so as to afford plenty of variety and are they graded so that the pupil will approach the difficulties gradually?

g. Do the problems stimulate thought? For example, in algebra do they lead the pupil to consider whether or not more than one solution is possible? whether or not any solution is possible? and whether or not any change in statement leads to special solutions of particular interest? To take a simple illustration, the study of the family of problems included in the general case of finding two numbers whose sum is n offers an interesting range of possibilities such as the following: (1) suppose that the numbers are consecutive and $n = 9$; (2) suppose that they are consecutive odd numbers and $n = 16$; (3) in each of the preceding cases, suppose that $n = 15$; (4) suppose that one number is $-2\frac{1}{2}$, or 0, or 11, or 17, and that $n = 15$; (5) suppose that $n = 0$ and that the numbers are both positive, are both negative, or are equal but have opposite signs.

h. Do the applied problems provide for the brilliant as well as for the dull pupil? for the rapid as well as for the slow pupil?

2. *Reviews.* In no subject is a pupil expected to keep in mind all that he has learned. To provide for retaining the essentials it is necessary that a textbook should contain a reasonable number of carefully selected reviews. A satisfactory test of the book, with respect to this feature, is provided in the following list of questions:

a. Is there a sufficient number of reviews of the right kind?

b. Do these reviews emphasize the fundamental parts of each unit of work?

c. Are the reviews properly placed?

d. Do the reviews encourage the pupil to organize his knowledge, selecting the important features, observing the connection of one topic with another, and building up the structure of the material as a whole instead of thinking of it as made up of detached parts?

e. To this end, are certain reviews cumulative, touching upon all preceding work and keeping the pupil refreshed as to all that has been studied before?

3. *Drill.* Besides the consideration of subject matter and reviews there is the equally important problem of drill work, not merely in algebra but in all branches of mathematics. This leads to the following questions:

a. Is there a sufficient amount of drill material and is it properly distributed in the textbook? No textbook can, within reasonable limits of space, provide for all needed drills and at the same time contain all the testing material necessary for maintaining the highest efficiency, but it can reasonably be expected to contain enough drill to prepare for such modern tests as are available for school use.

b. Is the drill material progressively graded so as to allow for the increase in difficulty in certain parts of algebra?

c. Is the purpose of each drill lesson made clear, so that the pupil may see the advantage of devoting his time to doing the work as well as he is able?

d. Does the drill work provide for the individual differences of the pupils, so that an assignment may increase or decrease in difficulty or in time allotment according to their needs?

e. In the interest of economy in learning, do the drill exercises cover systematically the various possibilities of error in the subject under consideration?

4. *Tests.* The rapid development of the testing movement has made it necessary for the schools to give careful attention to their construction. Tests are so important that the following questions may properly be considered in this connection:

a. Does each test meet the standard requirements of (1) validity, (2) reliability, (3) objectivity, and (4) comprehensiveness, as explained on pages 342 and 343 of this book?

b. Are the tests sufficiently numerous to be given with the necessary frequency?

c. Are a reasonable number of the tests suitable for use as teaching devices as well as measuring instruments?

d. Are the tests self-testing? (See page 343.)

e. Do they afford opportunities for making all possible errors under each topic, thus diagnosing the causes of the pupil's difficulties? In other words, are the tests diagnostic?

f. Do they train in habits of speed as well as accuracy? That is, are some of them so devised as to be used as timed tests?

Aids in Instruction. Not only is a textbook expected to supply subject matter, to arrange it psychologically, to select purposeful problems, to provide reviews and drills, and to prepare for modern tests, but it is expected to serve as an aid to the teacher in giving instruction, and to the pupil in receiving it. These two qualities should, therefore, demand our attention.

1. *As an aid to the pupil.* The following questions will assist the teacher in judging the textbook as an aid to the pupil:

a. Does it begin by making the pupil feel at home in the subject and by setting him thinking about its significance?

b. Does it give the pupil something to do as early as possible?

c. Are the explanations brief, clear, and "to the point"? Are they such as the pupils themselves can readily amplify when they are discussed? The alternative is that they shall be tedious in style and discouraging in length, a shortcoming that should condemn any book.

d. Does the book create an early appreciation of the value and beauties of the subject and a desire to pursue its study?

e. Does it cultivate a desire for investigation, so that pupils will be encouraged to go farther in the subject and in its applications to all branches of science and various phases of commerce and industry?

f. Does the textbook lead the pupil to do some real, constructive thinking; or does it, on the contrary, give the impression that algebra is only a piece of mechanism and that geometry is a subject for mere memorizing?

g. Does it aid in developing a reasonable attitude of mind toward problem solving, whether in algebra or any other branch to which it is related?

h. Does it encourage the pupil to help himself rather than continually to depend upon the book or the teacher?

i. In particular, does it give to the pupil plenty of opportunity for discovering by himself the principles of the science?

j. Are suggestions given to the pupil for expressing his work in succinct form, and is neatness encouraged?

k. Do the illustrative examples offer models which you would wish the pupils to follow in their written work?

l. Do the illustrations aid in the understanding of the work? Are they valuable object lessons to the pupils for making their own drawings?

2. *As an aid to the teacher.* In considering the textbook as an aid to the teacher the following questions will be found useful:

a. Does the book contain helpful suggestions to the teacher and are they conveniently placed?

b. Is there a full and well-arranged index which permits of easily finding the special topics?

c. Is the table of contents sufficiently extended to show clearly the outline of the work?

d. Are all the necessary tables and other similar items of information given in the body of the text or at the end?

e. Is the number of applied problems, of abstract exercises, and of drill pages so extensive that the teacher will not, under ordinary circumstances, be called upon to supplement them any further than by a good set of modern tests?

f. Do the explanations furnish material for brief discussions in class as to the best way of proceeding in the solution of algebraic exercises and problems of all types?

The Body, Language, and Soul of the Book. It has often been observed that it is the intangibles which really count in our lives, and it is so with respect to a book. There are certain things that we can feel but cannot always measure. The following paragraphs have to do with qualities of this nature, some of which are easily measurable and others not.

1. *The body of the book.* As to the mechanical make-up of a textbook we are in a field in which we can, if we choose, apply a figurative yardstick, as suggested by the following questions:

a. Is the book generally attractive in external appearance? Is the binding pleasing in all respects?

b. Is each page a model of neatness? Is the material so arranged that the page is attractive?

c. Is the type clear, of the proper size, well spaced, and such as to avoid the danger of eyestrain?

d. Is the paper of good quality, strong, without undue gloss, and of a tint that is restful to the eye?

e. Do the illustrations fit properly into the general scheme? Are they clear and attractive? Are they in every respect the models that you would wish your pupils to copy?

f. Is the book a model of good craftsmanship, strongly built and artistic in every feature?

2. *The language of the book.* Less easily measured is the language of a book, — not the mere vocabulary or the grammatical constructions, but those intangible elements that go to make up the style which characterizes it. The following questions suggest the features to be considered:

a. Has the book such dignity of language and a style so concise and clear as to command the respect and appreciation of the pupils and to serve as a model in making their own work refined and attractive?

b. Are the discussions and the problems expressed in good English, the punctuation being correct and the phraseology suited to the pupils for whom the book is written?

c. Is the language characterized by simplicity so as to be clearly understood by all who read it, and is there no excessive burden placed upon the pupil by reason of a pedantic or unnecessarily labored and burdensome vocabulary?

3. *The soul of the book.* Still less easily measured is the spirit in which the book is written. The significance of this statement may be more clear if we consider the following questions:

a. Speaking figuratively, and allowably so, has the book a soul, — one that shines out so as to influence both the pupils

and the teacher? Though intangible, does the spirit of the book lead the pupil to make friends with it and enjoy the time spent in its company?

b. Can you weigh the soul of a friend, or gauge it by points on a machine-made scale? Then why should we try to measure the soul of a book, or the soul of an author who writes the material for the pages and gives them the spirit which they figuratively possess? If the book has not this better nature, if it is nothing but mechanical drudgery, if it seeks to be judged only by the amount of obsolete work or by the number of problems it sets, then it has no spirit and is unworthy of a place in the schools.

QUESTIONS AND TOPICS FOR DISCUSSION

1. State succinctly those parts of algebra that the world generally needs; that the world generally uses; that the world would generally use if it were taught.

2. How should you make sure that your pupils understand what algebra means?

3. What are the reasons for requiring that certain parts of algebra be generally taught to everyone?

4. What is the best argument for introducing at least a half year of algebra before Grade IX?

5. Give a list of words still used in teaching algebra in Grade IX, but which are no longer essential.

6. Give a list of six formulas suitable for Grade VII that are (1) interesting, like $A = \pi r^2$; (2) reasonally understandable, say in biology; (3) not too complicated, the less so the better; (4) capable of evaluation by the class; (5) such as to permit of deriving one or more other formulas from each; (6) taken from one or more books not directly upon mathematics.

7. Contrast the various ways of beginning algebra, stating the advantages and disadvantages of each.

8. Show how the great increase in high-school population in the present century has complicated the problem of a proper course in algebra.

9. Discuss the question of the relation of the "project method" to the model lesson on page 302.

10. Discuss certain useful ways of meeting the problem of individual differences of ability in algebra.

11. Give a list of those features in the treatment of quadratic equations that might be omitted in the junior high school without endangering (1) the needs in other fields when the pupil enters college; (2) the needs of educated people in general.

12. If elementary algebra and demonstrative geometry as traditionally taught are satisfactory as to technique and logic, why are they not suitable for introductory courses?

13. Discuss the assertion that the most retarded child in the entire school system is the gifted one.

14. Select a simple type of objective in algebra which you have never seen analyzed, and make a careful analysis of unit skills or groups of skills involved. Indicate where you think the possibility of transfer may reduce the number of separate skills.

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CHAPTER VII

THE TEACHING OF NUMERICAL TRIGONOMETRY

1. Purpose in Teaching Trigonometry

Historical View. As with most other subjects of study, trigonometry has had certain notable periods in its development. It was for a long time merely an adjunct to astronomy, and for this reason spherical trigonometry received quite as much attention as that which related to the plane triangle. In those days it was based chiefly on geometry, the theory was difficult, and the computations were laborious. With the advent of a more convenient symbolism in algebra, however, early in the seventeenth century, it became more of an analytic subject and there developed the theory of trigonometric equations and identities. In the same period the invention of logarithms (first announced in 1614) revolutionized the methods of computation and thus rendered the work much more simple and more readily adaptable to problems in geodesy in general and in simple surveying in particular. The development of experimental physics in the eighteenth century opened up a new field of applications, so that the science ceased to be simply an introduction to astronomy or an abstract science like the theory of equations.

At the beginning of the twentieth century, however, there was left in the subject such a large amount of abstract theory as to disqualify it, as it was then conceived, for general teaching to elementary pupils. Teachers were justified in saying that it was too hard to be considered as a required topic in the field of general education.

This was unfortunate, for there is no subject that gives us such a good idea of our infinitesimal nature in the midst of space as this, which shows the general scheme of indirect measure, — of measuring celestial distances, of "lassoing the stars."

Furthermore, in surveying and in elementary science, even in the early stages, simple trigonometric formulas are playing a rôle of increasing importance, and it is desirable that the first steps in numerical trigonometry should be taken by all high-school pupils.

Why Some Trigonometry should be Required. There is now a general agreement among those who have given the subject any serious thought that a unit of numerical trigonometry should be required of all pupils in our junior high schools or in the first year of our four-year high schools. This agreement has arisen from a clearer conception of the nature of the topic in relation to modern life.

The place for trigonometry is naturally in algebra, after the pupil has learned to use formulas, simple radicals, and the linear equation. It is easily connected with intuitive geometry under the treatment of the geometry of size. It offers to the pupil an opportunity of understanding what is meant by indirect measurement, in contrast to the direct operation where we are able to apply the measuring instrument directly to the length or height to be measured. It is one of the most useful as well as one of the most interesting branches of mathematics.

The reasons, therefore, for requiring some slight knowledge of the subject may be summarized as follows: (1) trigonometry, dealing as it does with symbols representing numbers, is a part of elementary algebra and forms one of the most interesting applications; (2) it can be readily simplified so as to be much easier than such topics as the factoring of trinomials, — much more important, and much more interesting; (3) it offers to the pupil an opportunity for understanding what is meant by indirect measure; (4) it has various uses in surveying, in mensuration in general, and in elementary science; (5) it does not need demonstrative geometry as a basis, and since it pertains to figures, it fits in well with intuitive geometry; (6) by giving a clearer conception of the nature of mathematics and its relation to modern life, it becomes a part of the equipment of the citizen, just like geography, elementary science, and the general principles of industry.

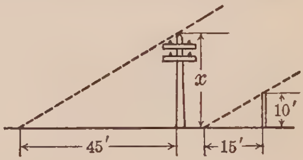
2. Content of the Course

What Part of Trigonometry meets this Requirement? The part of trigonometry about which all well-informed people need to know is that which shows how we measure distances indirectly; that is, how to determine the distance from where we stand or the height above our level without going to the place itself. It has nothing to do with intricate formulas but is concerned chiefly with very easy ones involving the sine, cosine, or tangent of an angle, and with simple numerical computations. On this account it is often called numerical trigonometry. As already stated, it is much simpler than most of the algebra that has recently been discarded or at least minified in the elementary courses. To teach it sufficiently well for the purposes in mind takes about three or four weeks, time that can easily be saved from unnecessary operations with certain types of polynomials, from unused cases in factoring, from certain manipulations of fractions, and from simultaneous equations with more than two unknowns.

Given this brief presentation a pupil will not be able to measure the distance to the moon or the sun, nor will he be prepared to find to a high degree of approximation the height of a mountain, but he will understand the elementary principles upon which these measurements are based and he will have acquired a type of knowledge that will make them seem more real and understandable. Such knowledge is quite as important as most of the facts that he will learn with respect to literature, history, geography, or the common sciences, and its acquisition can easily be made fully as interesting.

Meaning of Indirect Measurement. The pupil should be shown that we can make certain measurements directly by using a foot rule, a yardstick, a meter stick, or a steel tape, but that many measurements, such as the distance across a wide river, the height of a mountain, or the distance to the moon, must be made indirectly. He should be shown also that one way of making such a measurement is to compute it from the proportion that expresses the relation between certain sides of similar triangles.

For example, this figure shows a post 10 ft. high which casts a shadow 15 ft. long at the same time that a telegraph pole casts a shadow 45 ft. long. In finding the height of the telegraph pole, we have



$$\frac{x}{45} = \frac{10}{15}.$$

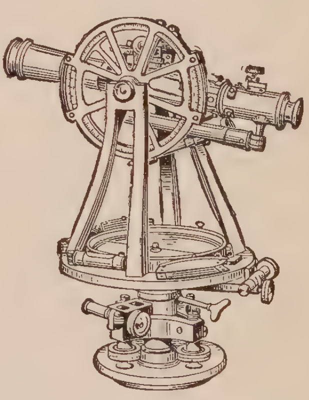
Solving, $15\,x = 450,$

and $x = 30;$

that is, the height of the telegraph pole is thus found to be 30 ft.

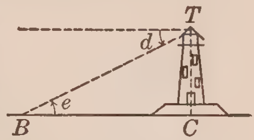
In all such work the pupil should understand that the objects are assumed to stand upright on level ground. He must also be taught that, as already stated, *no result of a computation can be more accurate than the least accurate of the measurements on which it is based*; that is, if we measure to the nearest 0.1 ft., our result can be correct only to the nearest 0.1 ft. He should also learn to use his common sense in deciding how far to carry a result under the given conditions.

Measuring Angles. The pupil should understand that, for measuring angles out of doors, surveyors or engineers use an instrument called a *transit.* If a transit is not available, a picture like the one given here may be shown. The circle at the left of the telescope is marked in degrees and half degrees, and from this scale is read the angle through which the telescope is moved up or down in sighting it at a distant object. The scale on the circle at the base of the supports of the telescope shows the angle through which the telescope is moved to the right or left.



A simple transit for school use can be made by using a small pipe and two cardboard protractors as described on page 364, and directions for constructing a homemade angle measurer are given on page 360.

Angles of Elevation and Depression. The pupil should be taught what is meant by the *angle of elevation* and the *angle of depression.* Thus, he can be shown that, in a figure like this, e is the angle of elevation of the top T of the lighthouse from a boat B, while d is the angle of depression of B from T. Care should be taken to see that the pupil understands clearly just what these angles are.



Scale Drawings. It is well to call the pupil's attention to the fact that one way of making an indirect measurement is to measure enough lines related to the required distance to enable us to draw the figure to scale, after which the required distance may be computed from the drawing, as shown on page 153. If no work in scale drawings has been done previously, it is well to consider a few examples of this kind before beginning the more formal work of numerical trigonometry.

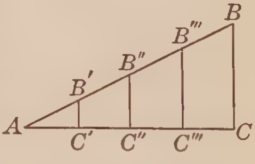
How to Begin the Teaching of Trigonometry. It is always desirable to start with a genuine problem, such as finding the height of the classroom. The teacher should choose some special angle of elevation, say 35°, so that the tangent is some simple number, in this case 0.7. This is important because a new idea should be acquired as easily as possible, and this can best be done by using simple numbers. Suppose, now, that the distance of the point of observation from the wall is found by measurement to be 20 ft. From their experience with scale drawings the pupils will know that the scale does not make any difference, but one row may be asked to take $\frac{1}{2}$ in. to represent 1 ft., and another row to take 2 cm. to represent 1 ft., and so on. The pupils should then all proceed to find the height, and they should all get approximately the same result.

Finding the Height of a Flagpole. It will then be interesting to ask the pupils to find the height of a flagpole or tree on the schoolgrounds. Take a point of observation such that the angle of elevation will be 35° and suppose that the distance of the point of observation from the foot of the pole is then found to be 100 ft. The less brilliant pupils will say, "Make a new drawing." The brighter pupils will know better; they will say, "It

is already made. The other problem gave us $\frac{7}{10}$, therefore the flagpole is $\frac{7}{10}$ of 100 ft., or 70 ft. high." The pupil should then be led to see that the height of the Woolworth Tower in New York City could be found by the same drawing.

Tangent as a Multiplier. Since the pupil cannot always expect the angle of elevation to be 35°, the class should be asked to look up the table showing what the fractions are for the various possible angles. This should be done before any further field work is undertaken. These fractions should be called *multipliers* at first rather than *tangents*. This is easily done, because in getting the height of the flagpole we multiplied 100 ft. by $\frac{7}{10}$, and the pupil sees at once what is meant by calling the ratio a multiplier. He has seen 3.14 used in the same way as a multiplier in the formula $\frac{C}{d} = \pi$, or 3.14. We may therefore state the problem thus, knowing the base of the right triangle: "What is the number which we must use as a multiplier to get the height?"

Tangent of an Angle. When the teacher wishes to introduce the term *tangent* and to refer to it as a ratio, it will be well to recall how, as described on page 220, the height of the telegraph pole was found by measuring the shadows and by using proportion. If the post in that case had been half as high, its shadow would have been only half as long; and if it had been twice as high, its shadow would have been twice as long. That is, the pupil should be shown that we are not so much concerned with the particular lengths of the post and the shadow as we are with the relation between these two lengths.

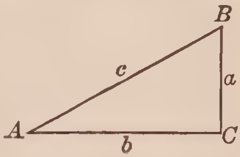


Similarly, if we consider BC in this figure as the height of the telegraph pole and $B'C'$, $B''C''$, and $B'''C'''$ as the heights of several different posts, we can easily see that

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''} = \frac{B'''C'''}{AC'''}$$

As we know, B' is read "B-prime," and, similarly, B'' is read "B-second," and B''' is read "B-third."

That is, the pupil should see that the ratios of the heights to the lengths of the shadows are constant as long as $\angle A$ is unchanged. Similarly, in this right triangle, he should see that for any given angle at A the ratio of a to b is constant, and he should be told that this ratio is called the *tangent* of $\angle A$, which we abbreviate by writing "tan A."



This figure shows the standard right triangle used in trigonometry in which we have the following important relations:

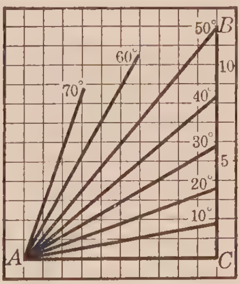
The tangent of either acute angle in a right triangle is the ratio of the opposite side to the adjacent side;

that is,
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b},$$

and
$$\tan B = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{b}{a}.$$

The pupil should now be told that in practical work we use tables in which the tangents are given for all necessary angles, these having been computed by methods of higher mathematics. The explanation of the use of such tables is given in any acceptable textbook on junior-high-school mathematics.

Finding Tangents. The teacher should help the pupil see how, if on squared paper we construct angles of various sizes at A, as shown in this figure, and if we then draw CB so that $AC = 10$, we can find approximate values for the tangents of the angles by taking $\frac{1}{10}$ of the number of spaces cut off on CB.



For example, we have

$$\tan 10° = \tfrac{1}{10} \times 1.8 = 0.18,$$

$$\tan 20° = \tfrac{1}{10} \times 3.6 = 0.36,$$

and so on for angles of different sizes.

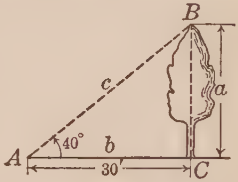
Using the Tangent. We should next show the pupil that since, as we have seen from the definition of tangent,
$$\tan A = \frac{a}{b},$$

we have
$$a = b \tan A,$$

which is a convenient formula for finding the value of a, since we have merely to multiply b by $\tan A$.

For example, if from a point 30.0 ft. from a tree the angle of elevation of the top is 40°, how high is the tree?

From the table of tangents we find that $\tan 40° = 0.8391$, and so $a = b \tan A = 30.0 \times 0.8391 = 25.1730$; that is, to the nearest 0.1 ft. the height is 25.2 ft. The pupil should be led to see that in giving the result in this way we take b as measured to the nearest 0.1 ft., it being given as 30.0 ft. The result is therefore given to the same number of places.



Sine of an Angle. It will not be difficult for the pupil to understand that, in the standard right triangle shown on page 223, there are other multipliers besides the tangent; for example, we may have $\frac{a}{c}$. This multiplier is called the sine of A, which is abbreviated by writing "sin A." Considered as a ratio, we have the following relation:

The sine of either acute angle in a right triangle is the ratio of the opposite side to the hypotenuse;

that is, $$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c},$$

and $$\sin B = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{b}{c}.$$

The table of sines should now be explained as in the case of the table of tangents (page 223). For example, from such a table we find that $\sin 37° = 0.6018$.

Using the Sine. Since, from the definition of sine,

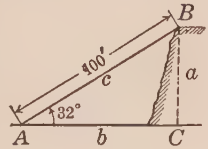
$$\sin A = \frac{a}{c},$$

we have $$a = c \sin A,$$

which is a convenient formula for finding a when it is easier to measure c and $\angle A$ than to measure b and $\angle B$.

For example, in finding the height of a cliff, some boys could not drop a plumb line from the top, and so they stretched a

100-foot tape from the top to the point A as here shown. If from A the angle of elevation of the top was then found to be 32°, what is the height of the cliff?



From a table of sines, $\sin 32° = 0.5299$, and so $a = c \sin A = 100 \times 0.5299 = 52.9900$; that is, to the nearest 1 ft., which is close enough with such approximate methods of measuring, the height of the cliff is 53 ft.

Cosine of an Angle. Another important multiplier for $\angle A$ in the standard right triangle shown on page 223 is $\frac{b}{c}$. This multiplier is called the cosine of $\angle A$, which is written "cos A." Considered as a ratio, we have the following relation:

The cosine of either acute angle in a right triangle is the ratio of the adjacent side to the hypotenuse;

that is, $$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c},$$

and $$\cos B = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{a}{c}.$$

The table of cosines should now be explained, as in the case of the tables of tangents and sines.

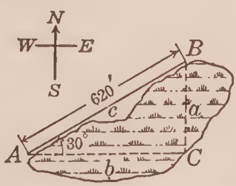
Using the Cosine. Since, from the definition of cosine,

$$\cos A = \frac{b}{c},$$

we have $$b = c \cos A,$$

which is a convenient formula for finding the value of b.

For example, suppose that a surveyor wishes to find the distance b on an east-and-west line across the marsh shown in this figure. Placing his transit at C, he sights northward, and his helper places a stake at B, from which they can measure the line AB. Moving his transit to A, he finds that $\angle A = 30°$ and that $AB = 620$ ft. What is the length of AC?



From a table of cosines, $\cos 30° = 0.8660$, and $b = c \cos A = 620 \times 0.8660 = 536.9200$; that is, to the nearest 0.1 ft. the required distance is found to be 536.9 ft.

Functions of an Angle. By this time the pupil will have been taught what it means to say that one quantity depends upon another for its value, and that any quantity which depends upon another for its value is called a *function* of the latter.

Since tan A, sin A, and cos A all depend for their values upon the size of $\angle A$, they are called *functions of the angle.*

These functions of an angle are called *trigonometric functions,* the word "trigonometry" itself coming from three Greek words which mean "three," "angle," and "measure." As we have already seen in the few cases that we have considered, a trigonometric solution depends upon measuring certain parts of a triangle. The work from here on should, however, be made a straightforward application of tangents, sines, and cosines rather than a complete and formal solution of a triangle in each case.

The functions that we have been using are also called *natural functions,* in order to help the pupil to distinguish them from other functions that are used in more advanced work.

Relation of Sine to Cosine. The pupil knows that if the sum of two angles is 90°, each is called the *complement* of the other; that is, in the trigonometric right triangle either acute angle is the complement of the other. Since $\sin B = \frac{b}{c}$ and $\cos A = \frac{b}{c}$, he should see that $\cos A = \sin B$; that is, the syllable "co-" is really an abbreviation for "complement," and the word "cosine" means "the complement's sine."

Subsequent Work. What has here been stated refers, of course, merely to the introduction to the subject. The subsequent work will naturally be regulated to a considerable extent by the textbook in use. Such a book should supply a sufficient number of interesting problems to show the pupil the practical uses of the three functions mentioned, and possibly, but not necessarily, of the cotangent. The work should be simple and should show the practical applications of the subject. If there is a transit available, it should be actually used out of doors, and not considered simply as an instrument to be talked about in class. If it is not available, a homemade substitute of the type suggested on pages 360 and 361 should, if possible, be employed, the prin-

ciple being the same as that of a more accurately constructed piece of apparatus. The teacher is advised to remember that a considerable number of simple problems, each having the elements of reality, accomplish the purpose in view much better than a small number of difficult ones. The latter, through their very difficulty, usually fail to arouse the pupils' interest and to reveal the objectives which should determine the details of teaching this particularly valuable branch of mathematics.

QUESTIONS AND TOPICS FOR DISCUSSION

1. In the nature of the subject what is there that is likely to make the study of numerical trigonometry of interest to pupils in the junior high school?

2. How and where in later life would a knowledge of numerical trigonometry be likely to be of use to a boy or girl?

3. Is there any reason why a boy should study trigonometry in the junior high school that does not hold equally well for a girl?

4. How can we best show the pupil the beauty of trigonometry as we show him the beauty of good literature, good music, and good painting?

5. Discuss the statement that trigonometry aids the engineer in calculating distances, the navigator in finding his position at sea, and the astronomer in exploring the heavens.

6. Select a list of ten new problems, suitable for teaching to pupils in the junior high school, which involve applications of the sine, cosine, and tangent.

7. Show how you would proceed to teach by means of scale drawings the problems selected in No. 6 above.

8. When will it be best to refer to the sine, cosine, and tangent as ratios and when as multipliers? What advantage, if any, has one over the other?

9. Write a lesson plan for teaching the first lesson on the tangent; on the sine; on the cosine.

10. Discuss the method you would use in teaching pupils to check the solutions to the problems in numerical trigonometry.

11. What should be the subject-matter requirements for a teacher of numerical trigonometry?

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CHAPTER VIII

THE TEACHING OF DEMONSTRATIVE GEOMETRY

1. Purpose in Teaching Demonstrative Geometry

A Mistaken View of the Subject. There is an impression that needs correction with respect to the purpose in teaching demonstrative geometry; that is, that we teach the subject because we need it in measuring fields or in finding the capacity of boxes and grain elevators, or that it is taught in order "to develop space perception." The fact is that we attain all these objectives quite as well by means of intuitive geometry; indeed, considering the time expended, we attain them rather better by such means.

If a pupil is not convinced that he can find the area of a rectangle before he studies demonstrative geometry, what he learns about it there will not convince him. All the common measurements that he will probably need, except in technical trades, are now taught him in any respectable course in the eighth grade or earlier. As to "space perception," which is a rather pompous phrase suggested by educational philosophers, he will secure this quite as well in a good course in trigonometry or in practical mensuration.

The Real Purpose. The real purpose of the subject is suggested by the word "demonstrative" rather than by "geometry." The mere utilities of geometry have already been acquired before the pupil begins, if he ever does, the work in what is to him an entirely new field, — that of logical proof. Nowhere in his previous training, nowhere else in his elementary education, does he come in close contact with a logical proof. The chief purpose of this part of mathematics, then, is to lead a pupil to understand what it is to demonstrate something, to prove a statement logically, to "stand upon the vantage ground of

truth." He sees a sequence of theorems built up into a logical system and he sees how this system is constructed, the result being a basis of proved statements which he can use for establishing further proofs, precisely as a lawyer proceeds to construct his case or a speaker to construct an argument. We use geometry rather than a course in pure logic because of the pupil's familiarity with the figures used in the former subject. These figures are made the basis of discussion.

Is it Worth While? There are writers on education who question the value of all this. Such educators have always existed. "What is the use of all this?" asked a pupil of Euclid. "Give him a penny," said the great geometer, referring to him as one who seeks money instead of seeking the truth.

People of this class would abolish history, saying that we should face the future instead of the past, — a phrase which catches the attention of those who think but little. They would abolish the study of the great literature of the past, saying that each generation should create its own classics, — as if classics could be made like cheap motor cars, by mass production. In the same way they would abolish all that does not relate directly to sociology, to economics, to religion, to politics, or to some other special line in which their own particular interests lie and which agrees with their particular brand of open-mindedness.

The question is simply this, Is it worth the while of a well-educated person to know what a logical proof means, to know how absolute truth can be found from certain data, to think rigorously in an abstract field? If so, then geometry offers the only material for the purpose that is usable in the secondary school, and it should be studied long enough and thoroughly enough to serve the purpose in view. If not, then demonstrative geometry must be relegated to the category of material needed only by those who wish to become mathematicians, either in the field of pure science or in the equally important fields of physics, astronomy, engineering, and the like. If we take this latter view, then we shall not need to require that all pupils know the meaning of a logical demonstration, and for the

same reason we shall no longer require them to know something of the various other great fields which make for broad knowledge.

To the mathematician his whole science rests upon the foundation of demonstrative geometry. Not until this subject is begun does the pupil really appreciate the significance of mathematics. It is here that rigorous logic begins to be applied; it is here that he first appreciates the step "If that, then this." Here he comprehends the significance of the further chain of reasoning illustrated by the statement "I can prove *A* if I can prove *B*; I can prove *B* if I can prove *C*; but I can prove *C*, and hence I can retrace my steps and prove *A*." By these two acquisitions he is inducted into the domain of mathematical thought, and this is the appropriate work of this unit.

Ability to Study Geometry. Probably the greatest contribution made in recent years to the science of education is to be found in the mental measurements of children. Perhaps it is the only great contribution. To tell by simple measurements of abilities the probable qualifications of a pupil for the study of any particular subject is, if we are certain of the validity of the method, a very great achievement. If we make a mistake in our prognosis, it is a matter for regret; but it is also a matter of regret if a pupil wastes his time in a hopeless struggle to do what he is mentally incapable of accomplishing.

The whole subject of mental tests is so new that we cannot say what degree of reliance we can place upon any one test which claims to say whether a pupil will succeed in a field which is entirely unknown to him. Just what new power he may develop if his interest is aroused, even though the preliminary tests failed to reveal it, we must hesitate to affirm.

Nevertheless it is certain that some minds are incapable of succeeding in a subject like demonstrative geometry. A knowledge of this fact, however, is probably more safely secured by the modern course in junior-high-school mathematics, in which there is a brief introduction to the meaning of a demonstration. If the pupil shows his inability here, it is rather safe to say that

he should be advised not to undertake further work in this field. If he is not capable of logical thinking, his chance for success lies in artistic, literary, or elementary mechanical fields, and he should be encouraged to proceed accordingly.

It would not be an evidence of good judgment, however, to exclude any pupil from the study of mathematics if he is eager to pursue it. Even if the results of the tests he has taken seem to indicate that he is not likely to succeed in the subject, the fact that he persists in his desire to proceed in the work is at least one good reason for allowing him to make the attempt. There are numerous instances of pupils who showed no promise in mathematics at first, but who became leaders in research later. Some of our greatest mathematicians did not discover their power to succeed in mathematics until they studied demonstrative geometry. Newton was a poor student until he took up geometry, and Einstein relates that his work in mathematics was of a low order until he became interested in the particular branch that made him famous. The possibility of latent power places a heavy responsibility upon the teacher to give every ambitious student a chance to know the subject that he professes to teach, and to teach it so that his pupils shall enjoy it and see its significance.

2. Outline of the Course in the Junior High School

Fundamental Propositions. We need only a few fundamental propositions for this unit of demonstrative geometry, beginning perhaps with those on congruence and ending with the theorem of Pythagoras. The fundamental propositions need only a few others as a basis. Several of those in the Euclidean treatment we now may accept in the junior high school without proof. Practically all of these are covered in the course in intuitive geometry. If we can agree upon a list of the things that we will accept in this way, and beyond that give only so many as are necessary to realize the purpose already stated, we shall greatly improve the present situation.

The difficulty with the ordinary geometry course in American

high schools is that we have held too strongly to the systematizing motive and have given the pupils an unadulterated form of Euclid. We should profit by the suggestions of the Mathematical Association of England,[1] even though we need not imitate in detail the practice therein advocated. The British list is not so long as the one formerly used in America, and the course is spread over a longer period of time, the result being that it is better understood. Our course must necessarily be somewhat uniform on account of the mobile character of our population, but it need not be as extensive as was formerly the case. We must keep in mind that the group entering the high school today is, on the whole, very different from that of thirty years ago and that their needs are not the same.

Number and Importance of Propositions. The results of geometry teaching can be improved by decreasing the number of theorems we teach, by concentrating on those propositions which are unquestionably fundamental, by recognizing the importance of original work, and by emphasizing the ability to demonstrate rather than the number of propositions studied and perhaps often memorized. In the tenth grade we now expect the pupil to master less than a hundred propositions.[2] One modern text has nearly a thousand original exercises, — a marked contrast to the lists of a generation ago.

There are teachers who feel that plane geometry cannot be taught in one year. This presupposes a "definite amount of ground" to be covered, which is purely a myth. We cannot teach all that is known of geometry in many years or even in a lifetime. We must remember that for some pupils a few propositions will suffice, while for others the number should be much larger. It therefore seems wise to make a careful selection for this unit of material in the junior high school and to do the best we can with it. Even though from the standpoint of the mathematician it is a good plan to give a more thorough and extensive knowledge of geometry than we ob-

[1] The Teaching of Geometry in Schools. The Mathematical Association. G. Bell and Sons, London, 1925.

[2] Revised Report on the Requirements in Plane Geometry, pp. 9–17. College Entrance Examination Board, 1923.

tain when we limit the field, nevertheless, when we consider the age of the pupils at present, and the age in which they live, we must recognize the fact that very legitimate claims can be advanced by other important branches of human knowledge which hardly existed in the nineteenth century, and that geometry can claim no monopoly.

The proper question for us to ask today is whether the legitimate claims of geometry as a method rather than as a body of facts can be met in less time than was formerly allowed, and the answer that will probably appeal to any unbiased mind is in the affirmative.

Extent of the Course. The extent of the course should be determined by two considerations: first, the length of time necessary to give the pupils a good idea of what it means to prove something; second, the time at the disposal of the school for such work. Experiment shows that in six weeks pupils can get a very good idea of what a demonstration means. When these same pupils later proceed to the formal study of geometry, their work shows the value of such a short introductory course. In any case, the teacher should be guided somewhat by the needs of his particular group.

How to Begin Demonstrative Geometry. It is often desirable to begin the work in demonstrative geometry in the junior high school by assuming the truth of the first congruence theorem, this assumption growing out of the intuitive work already done. Since the usual demonstration must necessarily rest on postulates only, it is legitimate to make this theorem itself a postulate if the immaturity of the pupil requires it. If this seems the better plan, we are justified in beginning the work by seeing how many exercises or other major theorems we can prove by its aid. The same method may be used with all the first three congruence theorems, either for the whole class or for the less gifted members. Logically, it is not an ideal plan although a justifiable one; psychologically, it is frequently the desirable method of procedure.

Basic Theorems for such a Course. Remembering that our objective in this introductory course is to show the meaning of

a demonstration, the following may be considered a desirable list of basic theorems:

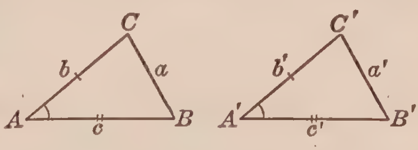
1. The usual congruence theorems.
2. The theorems relating to parallel lines.
3. The theorem relating to the sum of the angles of a triangle.
4. The theorems relating to similar triangles.
5. The Pythagorean Theorem.

With such an equipment alone, we could prove a large number of the original exercises of geometry. As to the Pythagorean Theorem, it can be proved with the aid of only a few theorems; indeed, it can be made the first theorem in a course, although the proof is then too difficult for most beginners.

It is unnecessary, in a book for teachers, to give proofs of all these basic theorems. We shall, however, discuss them all in some detail.

Congruent Figures. The pupil should understand that two figures having exactly the same shape and the same size are congruent. This means that they are identically equal in all their parts. Figures may have the same shape, as in the case of a large square and a small one, and not be congruent; they may also have the same size (area), as in the case of a triangle and a rectangle, and still not be congruent; but if they have both the same shape and the same size, then we have a case of congruence.

Study of a Figure. Having established the idea of congruence, it is an easy matter to lead the pupil to see that, as we look at these two triangles, in which $b = b'$, $c = c'$, and $\angle A = \angle A'$, we can easily infer that they are congruent. The question is, Is this inference correct, and are the figures necessarily congruent?



In studying a figure it is often helpful to mark the equal corresponding parts in some such way as here shown.

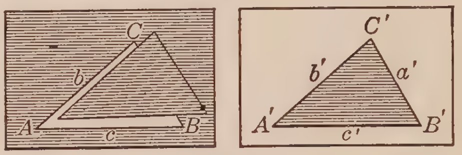
To answer this question, we proceed as follows:

If we wish to prove that two figures are congruent, what must we first show with respect to all the parts of the figures? What is the reason?

What effect does moving a triangle have upon its shape and size? What is the reason?

What way does this suggest of testing whether or not the parts of $\triangle ABC$ can be made to coincide with the corresponding parts of $\triangle A'B'C'$?

Suppose that we copy on a sheet of paper the $\triangle ABC$ shown above. Then cut out the triangle, as shown in the first of these figures, place it on a second sheet of paper, and draw pencil lines along the sides b and c, lettering these lines b' and c', and lettering the points where A, B, and C rest on the paper A', B', and C'. Then draw $B'C'$, or a'.



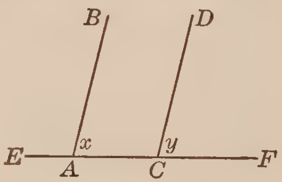
In these triangles we have reproduced the given conditions; that is, we have made $b' = b$, $c' = c$, and $\angle A' = \angle A$.

Do the triangles coincide in all their parts? Are they therefore congruent?

If you were asked to complete the statement "If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles . . .," how should you complete it?

The second congruence theorem can be similarly proved, after which the third becomes little more than a corollary to the first.

Parallel-Line Theorems. If in this figure the lines AB and CD have the same amount of angular rotation from the initial line EF, they have the same direction (meaning that they make equal angles, x and y, with EF, as shown) and are said to be *parallel*. The symbol for "parallel" is $\parallel$. Thus, AB is $\parallel$ to CD is read "AB is parallel to CD."



The angles x and y are called *corresponding angles*. The line EF is called a *transversal* of lines AB and CD. We can thus infer that the lines AB and CD are parallel only when the corresponding angles are equal, and that the corresponding angles are equal only when the lines are parallel. If our time is limited we do not

have to ask pupils to prove this. It is one of the basic theorems that can, in the junior high school, be assumed as true without proof. If this theorem is used as basic, the pupil should then be able to prove the following group of propositions associated with parallel lines cut by a transversal:

1. *If two parallel lines are cut by a transversal, the alternate angles are equal, and conversely.*

2. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary, and conversely.*

3. *If two straight lines are perpendicular to any given line, they are parallel.*

The pupil should see that the converse of the third of these theorems is not necessarily true, drawing a figure to illustrate his conclusion.

Sum of the Angles of a Triangle. We can now state and prove one of the most important theorems of plane geometry, as follows:

The sum of the interior angles of any triangle is 180°.

The equation growing out of the proof above is a very useful one as it enables us to find one angle of a triangle when the other two are known. Thus, if we know that two angles of a triangle are 50° and 70°, we see that the third angle must be 60°. This is of great practical value to the surveyor, who is thus enabled to find the size of all three angles of a triangle by measuring only two directly, or to check his work if he measures all three.

In the exercises that he meets, the pupil will need to apply all the theorems thus far mentioned.

Other Theorems easily Proved. The pupil will now find it easy to prove the following theorems:

1. *The sum of the three exterior angles of a triangle is 360°.*

2. *An exterior angle of a triangle is equal to the sum of the two nonadjacent interior angles.*

3. *The sum of the interior angles of a quadrilateral is 360°.*

4. *The sum of the exterior angles of a quadrilateral is 360°.*

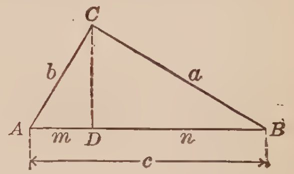
5. *Any two consecutive angles of a parallelogram are supplementary.*

6. *The opposite angles of a parallelogram are equal.*

7. *Two of the pairs of the consecutive angles of a trapezoid are supplementary.*

Pythagorean Theorem. If we have proved the theorems on similarity or have assumed that the sides of similar triangles are in proportion, it will now be easy to prove the well-known theorem of Pythagoras as follows:

1. In the figure here shown $\triangle ABC$ is a right triangle, C being the right angle. The pupil should first see that ⊿s ABC and ACD are similar, that $\frac{c}{b}=\frac{b}{m}$, and that as a result $b^2 = cm$.



2. He should then see that ⊿s ABC and BCD are similar, that $\frac{c}{a}=\frac{a}{n}$, and that as a result $a^2 = cn$.

3. He should then show by adding the corresponding members of these equations, that $a^2 + b^2 = cn + cm = c(n+m) = c^2$.

This theorem is one of the most important propositions in plane geometry and should be familiar to all pupils. It has many important applications.

3. Methods of Teaching

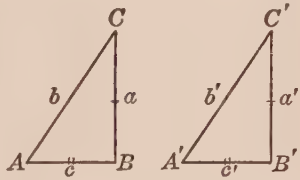
Using One Proposition to Prove Another. After we have accepted or proved a theorem like the one on page 236, we can show the pupil how to use it in proving other statements. That is, if from the given conditions about two triangles we can show that two sides and the included angle of one are equal respectively to two sides and the included angle of the other, we can state at once that the triangles are congruent by referring to the theorem as the reason.

This use of a proved theorem is illustrated by the following corollary about right triangles:

If the two sides forming the right angle of one right triangle are equal respectively to the corresponding sides of another right triangle, the triangles are congruent.

The discussion may be arranged as follows:

Given the $\triangle ABC$ and $A'B'C'$, right-angled at B and B' respectively, with $a = a'$ and $c = c'$.



***Prove that** $\triangle ABC$ and $A'B'C'$ are congruent.*

The plan is to show that this is a special case of the first congruence theorem.

Proof. $a = a'$,

and $c = c'$,

because these sides are given equal,

and also $\angle B = \angle B'$,

because all rt. $\angle$s are equal.

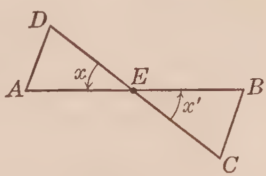
Hence $\triangle ABC$ and $A'B'C'$ are congruent,

because they have two sides and the included $\angle$ respectively equal.

In giving the reason for the last step we might have quoted the complete theorem. We shall frequently, however, find it desirable to abbreviate in this way in order to save copying long statements.

Graded Exercises. The teacher should observe that the corollary proved above really constitutes what may be called a one-step exercise. That is, the main step consists in realizing that the right angles are respectively equal and that the triangles are therefore congruent.

A similar case would arise if the pupil were asked to prove the proposition *If two straight lines bisect each other, and if their extremities are joined by straight lines, two congruent triangles are formed.* Here the one step consists in seeing that $x = x'$, the reason being that vertical angles are equal. It follows, then, that the triangles are congruent.



After a theorem has been discussed it is desirable to begin with simple one-step exercises like those above, and to proceed gradually to a consideration of what may be called two-step exercises, three-step exercises, and so on. An example of a two-step exercise will now be given.

Two-Step Exercise. If in the case just discussed the pupil had been asked to prove that $AD = BC$, the proof would have involved two steps. The first step is the statement that $x = x'$, from which the triangles are seen to be congruent; and the second step is the statement that $AD = BC$, because they are corresponding parts of congruent triangles.

In order to make clear what such a psychological order of treatment of original exercises signifies, we shall now illustrate what we mean by a three-step proof.

Three-Step Exercise. Let us suppose that we ask the pupils to prove the proposition

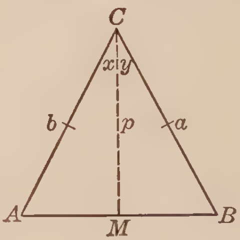
In an isosceles triangle the angles opposite the equal sides are equal.

We arrange the discussion in the usual form, as follows:

Given the isosceles $\triangle ABC$ with $a = b$.

Prove that $\angle A = \angle B$.

The plan is to make $\measuredangle A$ and B corresponding $\measuredangle$ of two congruent $\triangle$.



Proof. Let p be the bisector of $\angle C$, forming $\measuredangle x$ and y as shown.

An $\angle$ can be bisected, and by one and only one line.

Letting M be the point where p meets AB, we see that in $\triangle AMC$ and BCM $\qquad b = a,$

because these sides are given equal,

that $\qquad p$ is common to both $\triangle$,

and also that $\qquad x = y,$

because p bisects $\angle C$.

Then $\qquad \triangle AMC$ and BCM are congruent,

because they have two sides and the included $\angle$ respectively equal,

and hence $\qquad \angle A = \angle B,$

because they are corresponding $\measuredangle$ of congruent $\triangle$.

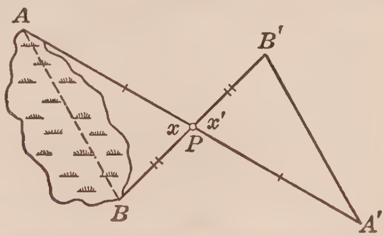
If the pupils have had the usual construction work, they can actually bisect $\angle C$ in the first step of the proof.

The three steps consist, first, in bisecting $\angle C$; second, in seeing that p is common to both triangles, from which fact the

triangles are congruent; and third, that $\angle A = \angle B$, because they are corresponding parts of congruent triangles.

The teacher should not make the mistake of thinking that one-step exercises are always easier than those requiring two steps or even three steps. The ease of proof depends in part upon the nature of the figure, and if this is complicated or unfamiliar in a one-step case, the proof may be more difficult than that of a three-step exercise requiring a simpler representation.

Practical Applications. It is always interesting to show the pupils how a given theorem may be applied in a practical way. For example, we may show how the first congruence theorem may be used in measuring distance in a case where it is not possible to measure the length of a line directly. For example, suppose that we wish to measure the distance AB across the swamp shown in this figure.



To do this we may place a stake at a convenient point P, and then by sighting from A we may locate a point A' on AP prolonged such that $PA' = PA$. In like manner, we may locate a point B' such that $PB' = PB$, and we may then find the length of AB by measuring $A'B'$.

The fact that $AB = A'B'$ is now easily proved from the first congruence theorem by showing that

$$PA = PA',$$

because these lines were constructed equal,

and that $$PB = PB',$$

for the same reason.

Designating $\measuredangle$ APB and $A'PB'$ as x and x', we then have

$$x = x',$$

because if two lines intersect, the vertical $\measuredangle$ are equal,

and hence $\triangle$ ABP and $A'B'P$ are congruent,

because they have two sides and the included $\angle$ respectively equal.

Therefore $$AB = A'B',$$

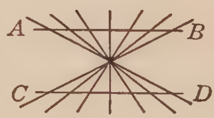
because they are corresponding sides of congruent $\triangle$.

In all cases like this, where the measurements of the distances are made out of doors, we assume that the land on which the measurements are made is a horizontal plane unless the contrary is stated.

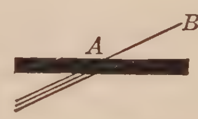
Use of Optical Illusions. It is sometimes the case with pupils who have studied intuitive geometry and have been permitted to take certain geometric truths for granted, that some of them will persist in saying that certain things are true because any-one can "see" that they are so. With such pupils it will be helpful to discuss a few interesting optical illusions. In this way they may be led to conclude that seeing is not always believing. In other words, the pupil should be led to see that the eye is a poor judge of facts, especially if it is not supported by some kind of check.

The following exercises and others of like nature will help a pupil to see that he must be careful in forming judgments merely from what he sees.

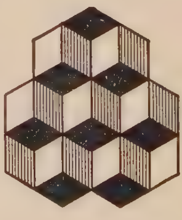
1. Look at this figure and state if AB and CD are both straight lines. If one of them is not straight, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.



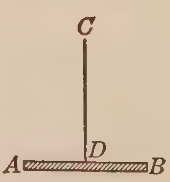
2. Look at this figure and state which of the three lower lines is BA prolonged. Then test your answer by laying a ruler along BA.



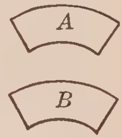
3. Thinking of the black portions as the upper faces of cubes piled in the corner of a room, study this figure and state how many complete cubes you see. Then thinking of the black portions as the bases of cubes, state how many complete cubes you see. If possible, explain the illusion.



4. In the figure here shown, which line seems to be the longer, AB or CD? If you think that CD is the longer, about how much longer does it seem to be than AB? After writing your answers, check them by measuring with a ruler.



5. Does the outline of this figure appear to form a square? If not, which is the longer, the vertical side or the horizontal one? After writing your answers, check them by measuring with a ruler.



6. Which of these figures appears to be the larger? Does it look longer or wider, or both? After writing your answers, trace figure A on thin paper, cut it out carefully, and place it over B, thus verifying your answers.

4. Fundamental Principles and Definitions

Bases of Proofs. The pupil should by this time begin to see that in contrast to intuitive geometry, or what is occasionally called experimental geometry, demonstrative geometry does not depend upon either observation or measurement. It is concerned chiefly with demonstrating that certain important statements are true and that they are arranged in some systematic and logical order. He should now be led to see that proofs of statements in geometry are based upon certain fundamental assumptions of obvious truths.

In his study of algebra the pupil used certain axioms in solving equations. He should continue to use these axioms as well as a few others if necessary. He should also understand the use of certain postulates, these being statements relating only to geometry, and so obvious as to be generally accepted without any attempt at proof. He should see that the postulates do not depend upon observation alone, but also upon common sense, and that they are accepted as part of the foundation upon which the subsequent work may be built.

The Number of Assumptions should be Small. The interest as well as the value of geometry lies to a considerable extent in the fact that from a small number of assumptions it is possible to deduce a practically unlimited number of conclusions. With the truth of these statements we are not so much concerned as with the reasoning by which we draw the conclusions, although it is evidently desirable that the assumptions should

not be manifestly false, and that they should be reasonably few. It is in the logical investigation of statements, and in the discovery of new truths based upon the assumptions, that the real value of geometry lies.

To select the "irreducible minimum" of assumptions, however, is to offer a set of statements quite unintelligible to pupils beginning geometry or any other branch of elementary mathematics. Such an effort is laudable when the results are intended for advanced students in the university, but it is merely suggestive to teachers rather than usable by pupils when it touches upon the primary steps of any science. In recent years several such attempts have been made, and the names of a number of leading mathematicians have been connected with them, but in the practical work with beginning classes they have no particular significance.

Axioms needed in Geometry. There are various axioms to be found in mathematics. Out of these, the following are the ones that are essential in beginning geometry:

1. *If equals are added to equals, the sums are equal.*
2. *If equals are subtracted from equals, the differences are equal.*
3. *If equals are multiplied by equals, the products are equal.*
4. *If equals are divided by equals, the quotients are equal.*

The divisor must never be zero, because division by zero has no meaning.

5. *A number or magnitude may be substituted for its equal.*

For example, if $a + x = b$ and $x = y$, then $a + y = b$; and if $b > x$ and $x = y$, then $b > y$.

In geometry the term *magnitude* includes solids, surfaces, and lines.

As a special case of this axiom we often say that

Quantities equal to the same quantity are equal to each other.

6. *Like powers or like roots of equal numbers are equal.*

Since raising a number to a power is a case of multiplying the number by itself, and taking a root may be regarded as dividing by equal factors, this axiom follows from Axioms 3 and 4.

It will be observed that common sense tells us that the above statements must be true, and we shall therefore assume them to be true without proof. Other axioms may be assumed if and when they are needed.

Teachers should understand that the axioms are to be applied in a common-sense way. For example, when we quote Axiom 6, we do not mean that we may select the positive square root in one case and the negative one in another. That is, because $4 = 4$ and $\sqrt{4} = +2$ or -2, it does not follow that $+2 = -2$. It simply follows that the "principal roots" are equal; in this case these are the positive real roots.

Postulates needed in Geometry. The first four of the following postulates embody general rules of procedure that we shall use frequently in proving statements, and the others state important geometric facts which common sense tells us that we may accept as true without any proof:

1. *One straight line and only one can be drawn through two points.*

This postulate is sometimes more conveniently expressed in one of the following forms:

Two points determine a line.

Two straight lines cannot intersect at more than one point.

The truth of this last statement is apparent from the fact that if the lines should intersect at two or more points, they would coincide.

2. *A straight-line segment can be produced to any required length.*

When we speak of producing the line segment AB, we mean that the segment is to be extended through B. If we extend the segment through A, we should be producing the line segment BA.

3. *In a plane one and only one circle can be drawn with any given point as center and any given line segment as radius.*

Some writers use the expression "describe a circle" instead of "draw a circle."

4. *Any figure can be moved without altering its shape or size.*

That is, since we can draw the same figure in any number of different positions, we may think of it as being moved without any change in shape or size.

5. *A straight-line segment is the shortest path between two points.*

Since distance in a plane is measured along a straight line, this postulate is sometimes stated as follows:

A straight line is the shortest distance between two points.

Speaking precisely, however, distance is the *length* of the line.

6. *All straight angles are equal, and all right angles are equal.*

This postulate follows from the definition of a straight angle and of a right angle.

7. *Angles which have equal complements or equal supplements are equal.*

8. *A line segment can be bisected, and in one and only one point.*

At first the pupil may show the reasonableness of this postulate by drawing a figure. This statement can later be proved if desired.

9. *An angle can be bisected, and by one and only one line.*

As in Postulate 8, the pupil should at first draw a figure to show that this is true. The statement can be proved later if desired.

10. *There is one and only one line which, passing through a given point, is perpendicular to a given line.*

At first the pupil may draw the perpendiculars by means of a draftsman's triangle. He should see the apparent truth of the statement that at the point P on the line l only one perpendicular to l can be drawn, and similarly that from the point Q only one perpendicular to l can be drawn. He can prove later that these statements are true.

As already noted, certain of the above postulates of geometry are so simple as to admit of no proof, while others can be proved if we wish to do so. With pupils who are just beginning demonstrative geometry it is advisable to assume a number of postulates that can later be proved. For example, if we should try to prove that a line segment can be bisected, and in one and only one point, a pupil would feel that the statement is so evident that it is not worth proving. It is only after he sees the necessity for constructions that he feels any need for demonstrating the truth of the statement that a line segment can be bisected at all.

There are other postulates of geometry, the most important one being that through a point outside a given line one and only one line can be drawn parallel to that line. This is better postponed until the subject of parallels is considered. There is also a postulate which is tacitly assumed in geometry, namely, that if a straight line cuts a circle or other closed figure once, it must, if produced far enough, cut it again. There are other assumptions of this kind, such as that if two circles intersect once, they intersect twice, but it is generally agreed that such postulates

may be tacitly assumed. If a pupil feels that they should be included, he should be told that he ought then to add them to his list.

If the teacher will consider the nature of space, as discussed on page 308, he will find that certain axioms and postulates are not true in other spaces than the one with which we are most familiar, and which we may imagine to exist. These considerations will not, however, modify the presentation of geometry to beginning pupils. As we proceed in any science our horizon broadens and various definitions and assumptions broaden accordingly.

Nature and Purpose of Definitions. When we consider the nature of geometry it is evident that more attention must be paid to accuracy of definitions than is the case in most of the other sciences. The essence of all geometry worthy of serious study is not the knowledge of some fact but the proof of that fact; and this proof is always based upon preceding proofs, assumptions (axioms or postulates), or definitions. If we are to prove that one line is perpendicular to another, it is essential that we have an exact definition of "perpendicular," else we shall not know when we have reached the conclusion of the proof.

As stated on page 251 the essential features of a definition are that the term defined shall be described in words that are simpler than, or at least better known than, the term itself; that this shall be done in such a way as to limit the term to the thing defined; and that the description shall be as concise as seems reasonable, considering the abilities of the pupils. It would not be a good definition to say that "a right angle is one fourth of 360° and one half of a straight angle," because neither of the concepts "360°" and "straight angle" is so simple as the concept "right angle," and because the definition is redundant, containing more than is necessary.

Definitions not always Possible. It is evident that satisfactory definitions are not always possible; for since the number of words in our language is limited, there must be at least one term that is as simple as any other, and this cannot be described in terms simpler than itself. Such, for example, is the term

"angle." We can easily explain the meaning of this word, and we can make the concept clear, but this must be done by a certain amount of circumlocution and explanation, not by a concise and perfect definition. This explanation may be given in various ways, such as referring to the hands of a clock or gradually opening a pair of compasses, or by beginning with an arc of a circle and the corresponding central angle. Unless a beginner in geometry knows what an angle is before he reads the definition in a textbook, however, he will not know it from the definition itself.

We should always remember that a definition should be the outgrowth of previous or present experience and should not be looked upon merely as a basis for work. For example, never to have used and understood such terms as *right angle*, *perpendicular*, and *circle* before reading definitions of these terms would be an evidence of very poor teaching.

Existence of Geometric Concepts. It should also be understood in this connection that a definition makes no assertion as to the existence of the thing defined. If we say that a tangent to a circle is an unlimited straight line that touches the circle in one point, and only one, we do not assert that it is possible to have such a line; that is a matter for proof. Not in all cases, however, can this proof be given, as for the existence of the simplest concepts. We cannot, for example, prove that a point or a straight line exists after we have made a somewhat futile attempt to define these concepts. We therefore tacitly or explicitly assume (postulate) the existence of these fundamentals of geometry. On the other hand, we can prove that a tangent exists, and this may properly be considered a legitimate proposition or corollary of elementary geometry. In relation to geometric proof it is necessary to bear in mind, therefore, that we are permitted to define any term we please; for example, "a seven-edged polyhedron" or a "ten-faced regular polyhedron," neither of which exists. Strictly speaking, however, we have no right to use a definition in a proof until we have shown or postulated, tacitly (as in the case of such elementary concepts as a point or a line) or formally, that the thing defined has an existence.

Definitions to be Learned. In our daily lives we are continually using terms that we cannot readily define. This is perfectly proper, there being no reason for committing a dictionary to memory. We should all have difficulty in defining such everyday terms as "air," "gas," "fire," and "number." We could make an attempt and could use a number of words in so doing, but the definitions would not be particularly helpful to a foreigner who wished to understand the terms. Similarly, in geometry, certain terms like *straight line, space, angle,* and *solid* cannot be satisfactorily defined, although we can use them properly.

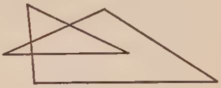
The test to be applied to requiring the memorizing of a definition in geometry is this: Does the pupil need to refer to the definition in proving a proposition? If so, it will be convenient for him to memorize that definition. If he is not going to use it in a proof, then he need not memorize it. As examples of terms of which the definition is not to be used in a proof, we have such words as *curve, ray, angle,* and *straight line.* As examples of terms that are to be so used and which should therefore be memorized, we have such words as *bisector, right angle, perpendicular,* and *circle.*

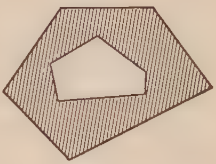
Any textbook that is carefully written will give instructions as to what definitions need be memorized, and will reduce the number as much as possible.

It is desirable to introduce the definitions at the points where they are needed. There are several that are needed even before the first proposition is proved, but as a general principle it may be repeated that definitions should be the outgrowth of experience (as of propositions previously proved), and should be given immediately, or at least shortly, before the proposition in which they are to be used.

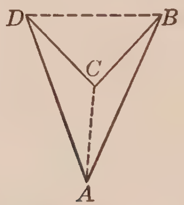
Precision of a Definition. It sounds well to say that every statement in geometry should be precise and that every definition should be so clear as to anticipate every question that may arise concerning the term defined. Every teacher knows, however, that precision of definition is difficult and that precision of measurement is impossible. We can have a high degree of precision, as with a watch or a ruler, but absolute precision of

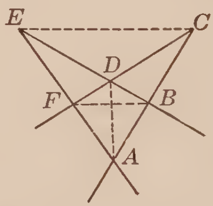
mechanical instruments for measuring is an impossibility. If the teacher feels that absolute accuracy of definition should be required of a pupil, let him first define a polygon and then decide whether these figures come within his definition. If this offers no difficulty, let him decide upon the smallest number of vertices that a polygon may have; and having done so let him ask himself whether a regular digon (two-angled polygon) is possible, and if not, why not. Let him further inquire whether the diagonals of a quadrilateral lie inside or outside the figure, and consider such a simple case as is shown in the first figure below, where the diagonals of the quadrilateral are *AC* and *BD*.





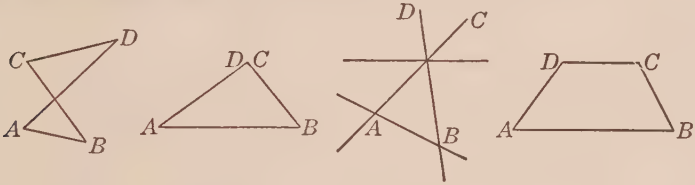
And finally, let him consider whether a quadrilateral is formed by four intersecting lines, and how many diagonals are possible, and then see if his statements meet all the conditions suggested by the second figure shown below.





Such questions should not be introduced in the early stages of geometry. At that time a pupil may properly think of a quadrilateral as convex, as in this figure. This figure, as he properly conceives it, has only two diagonals, as shown by the dotted lines. The teacher, however, should realize that in general a quadrilateral is formed by four straight lines lying in a plane. If we adopt the modern phraseology we say that they intersect in points at a finite distance or at an infinite distance, and that in any case we have three diagonals, even though some may be infinitely far away. For example,

these simple cases may profitably be considered by the teacher and perhaps referred to later in the pupil's course:



Such considerations may suggest more forcefully to the teacher the desirability of not committing to memory definitions that are not actually used in proofs, since we are constantly extending our ideas of even very common terms, and any claim for absolute precision of definition is quite sure to be a hindrance to progress and to be withdrawn as we proceed in our work.

Suggestions respecting Definitions. The following suggestions are in accord with the spirit of the preceding statements:

1. *Memorize no definition that is not soon to be used in some form of mathematical proof.* For example, we need to use the term "perpendicular" in geometric proofs, but not the term "angle." We explain the latter, because we talk about it; we define the former, because we need to say, "therefore a is perpendicular to b by *definition*." Similarly, we explain such terms as "line," "point," and "surface," but we do not need definitions of these terms in any proof.

2. *Reject all definitions in which the words used to define a term are not simpler than the term defined.* For example, to say that "an angle is the difference in direction of two lines" is unsatisfactory, because "difference of direction" and "direction" itself are more difficult to define than "angle."

3. *Never formally define a term until the pupil knows what it means.* No pupil gets his idea of the meaning of triangle from a formal definition; he has it long before he sees any such statement. Moreover, since he is not going to use the formal definition of triangle as authority for any statement in a proof, there is no good reason for his learning the precise words of some textbook writer respecting it.

4. *Never use a definition that is not simply expressed.* Do not be led away by the idea that the definition must be so strictly logical that it need never be changed. Remember, for example, that the old-time definition of multiplication crumbles to pieces as soon as we come to multiply by a fraction, by zero, by a negative number, by 2, and by -1, and that any definition that would cover all these cases would be absolutely unintelligible to all pupils in the first ten grades, and to most of their teachers.

No great writer seeks to use difficult words with which to express his thoughts; in technical fields he may be compelled to use them, but he endeavors to write in simple language. Similarly, no great teacher seeks to make his explanations, definitions, and proofs appear learned by the use of difficult words; he seeks to express himself in the language of his pupil.

General View of Assumptions. We are now in a position to take a more general view of the assumptions and definitions of geometry. Human liberty may lead us to say, "I shall define a straight line as crooked," and "I shall assume that the results of adding equals to equals are always unequal." If we do this, we can proceed and see what results will follow. Since we very soon get tangled in a mass of absurdities, we are usually glad to abandon our ideas of independence and come back to what the world generally calls a state of common sense. This common sense leads us to say that definitions and axioms are general agreements as to the meaning of words that we use and as to what we assume to be fundamental truths in any science.

In mathematics, therefore, we have several types of assumptions:

1. A definition is an assumption as to the meaning of some word.

2. An axiom in mathematics is an assumption of the truth of some general mathematical statement.

3. A postulate in geometry is an assumption of the truth of some geometric statement.

We might call all three simply "assumptions," but it is more convenient to make use of the three distinct terms.

5. Purpose of Basal Propositions

Distinction between Theorems and Problems. The theorems of geometry are concerned with proving geometric statements; the problems are concerned with the construction of geometric figures in a plane, the only instruments allowed being an unmarked straightedge and a pair of compasses. In solid geometry we assume that the necessary figures can be constructed, and so we do not attempt to show how this can be done.

In early days, upwards of two thousand years ago, the writers on plane geometry did not attempt to prove any theorem until they had shown that the figure could be constructed. For this purpose they placed some of their problems of plane geometry first, introducing others as needed. At present we generally assume that all figures in plane geometry can be constructed, as we assume it for solids, leaving until a later time the proof that a number of important constructions can be effected.

We might go even farther, for it would be possible and logical to assume all constructions in plane geometry, just as we assume them in more advanced work. We do this in some cases as it is; for example, we would not hesitate to ask a student to tell the number of right angles in the sum of the interior angles of a regular seven-sided polygon, although it is impossible to construct such a figure with the limitations imposed upon plane geometry, namely, of using only the compasses and the straightedge.

Because of this modern view of the case, we generally place the problems at the end of any particular book or chapter, although they might, as with ancient writers, be scattered among the theorems.

Model Proofs. The question now arises as to how proofs should be presented. Should we give them in full? Should we give them in full at first and gradually leave more and more gaps for the pupils to fill so as to make the proofs complete? Should we dictate the propositions and have the pupils work out the proofs? Should we follow a syllabus instead of dictating, still leaving the proofs to be worked out? Should we em-

ploy intuition and pretend to discover the propositions, and then invent our own proofs, the entire class perhaps working them out with merely the guidance of the teacher? Should we give suggested proofs, the pupils following out the suggestions, we ourselves pretending to encourage an originality which the suggestions render impossible? Or should we make some other combination or experiment, knowing very well that the same thing has doubtless been attempted many hundreds or thousands of times before? Given enthusiasm and personal magnetism, any one of these plans will yield fairly good results; a first-class teacher can succeed with almost any subject and with almost any plan of teaching it.

The plans are not equally good, however, and world experience has generally favored the use of a textbook that gives the proofs of the early propositions in full, gradually reducing the degree of completeness and leaving the pupil more and more upon his own responsibility in completing the demonstration.

The purpose in giving a complete proof at first, in clear style, with the reasons for each step stated in full form, is that the pupil may have a model before him. The reason for giving substantially complete or only partial proofs thereafter is that an approximate model may be before him even after he has come to rely more fully upon himself. It should never be assumed that proofs are given only to be memorized; they are given in order that a student shall have, every day or two, a model for his treatment of the all-important exercises, these constituting the field in which his originality, his insight into geometry, and his ability to think logically are to be shown.

Whether the student writes his reasons under each statement of the proof or at the right is a matter of little moment. In the printed page a larger type and a much clearer arrangement can be used if the reasons follow the steps, but in written work on a wide sheet it is quite allowable to place them at the right, and many teachers prefer this because they find that they can the more readily mark the paper when so written.

In any case, the model in the textbook will serve to keep before the student the necessity for succinct and logical expression.

6. Treatment of the Propositions

Misuse of the Blackboard. The question of the proof of the basal propositions demands first of all that we consider the use of the blackboard. In general the European schools have small boards, whereas in our country it is not unusual to see rooms in which they extend around the four sides. This is especially noticeable in some of our older buildings.

It is probably true that in no other way have we wasted time and cultivated bad habits in geometry as we have through the misuse of our blackboards. To send a whole class to the board to draw the figures and to write out the proofs is to waste time and encourage dawdling. It is impossible to have rapid, energetic thought or expression by any such plan.

To send a single pupil to the board, there to draw the figure and then to give the demonstration is not quite so bad, but it is usually bad enough. If the figure is at all complicated, the time required in drawing it with any degree of accuracy is substantially wasted so far as the rest of the class are concerned. It is much better to have large pieces of cardboard upon which neat and clear drawings have been made with black crayon, these drawings illustrating respectively the most important of the theorems, and preferably being the same as the ones given in the textbook. In this way no time is lost in placing the figures before the class, and the book proof can then be discussed by the whole class with reference to the figure given in the text. The use of a revolving circular blackboard, which places figures in different positions, is discussed on page 366.

When a pupil is explaining a proposition, do not let the proof be obscured by his speaking of angles as *ABC*, *BCD*, and so on, because in such a case the class will always have difficulty in following the demonstration. The work becomes much clearer if the student points to the angles and lines, saying, "This angle equals that because . . . ," or "This line equals that because . . . ," thus encouraging the class to follow easily the demonstration. We should continually emphasize the "chalk and talk" method in such a procedure.

Discussion of Proofs. While considering the model proofs in the book, however, many teachers obtain the most satisfactory results by rarely using the blackboard at all, reserving it for the exercises. The pupils open their books and each step is discussed, the statements being given rapidly, together with the reasons for each one. In the first few propositions these reasons are given in the text, but it often becomes necessary later to look them up from references and to supply any steps that are missing. The work is quite comparable to the silent-reading lessons that are rather common at the present time, — lessons which are first read and are then rapidly discussed. When this is done with the same spirit that is shown in oral work in algebra and arithmetic, the results are very satisfactory. If, however, the pupils are allowed to hesitate, to talk as slowly as possible, and to waste time generally, the method loses value and lacks all interest. If one pupil is asked to read a step to himself and then to give rapidly the reason, and another is asked to do the same for the next step, and so on, the work soon comes to be as enthusiastically done as it is in an algebra drill. This plan allows for board work with respect to any special points in the proof, but such work should be done rapidly and with interest.

7. General Methods of Attack

No Single Method for All Cases. There is no single method that is applicable to every exercise in geometry, and this is a fortunate fact. If it were not so, the attack would be too mechanical to be interesting. There is no one rule for solving every problem nor even for seeing how to begin. On the other hand, a pupil is saved some time by having his attention called to a few rather definite lines of attack, and he will undoubtedly fare the better by not wasting his energies over attempts that are in advance doomed to failure.

Two General Questions. There are two general questions to be considered: first, as to the discovery of new truths, and second, as to the proof. With the first the pupil will have little to do, not having as yet arrived at this stage in his progress. A

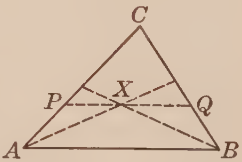
bright student may take a little interest in seeing what he can find out that is new (at least to him), and if so, he may be told that many new propositions have been discovered by the accurate drawing of figures; that some have been found by actually weighing pieces of sheet metal of certain sizes; and that still others have made themselves known through paper folding. In all of these cases, however, the supposed proposition must be proved before it can be scientifically accepted, although we may be perfectly convinced of its validity before we attempt any demonstration.

The Synthetic Method of Attack. As to the proof, the pupil usually wanders about more or less until he strikes the right line, and then he follows this to the conclusion. He should not be blamed for doing this, for he is pursuing the method that the world followed in the earliest times, and one that has always been common and always will be. This is generally spoken of as the synthetic method, the building up of the proof from propositions previously proved. It is, therefore, a method of proof rather than one of attack. If the proposition is a theorem, it is usually not difficult to recall propositions that may lead to the demonstration, and to select the ones that are really needed. If it is a problem, it is usually easy to look ahead in the proposed solution, to see what is necessary for its accomplishment, and to select the preceding propositions accordingly. In either case, it is a process of looking about and of recalling other related propositions, this being followed by a synthetic ("putting together") proof.

The Analytic Method. But pupils should be told that if they do not rather easily find the necessary propositions for the construction or the proof, they should not delay in resorting to another and more systematic method. This is known as the method of analysis, and it is applicable both to theorems and to problems. It has several forms, but it is of little service to a pupil to have these differentiated, and it suffices that he be given the essential feature of all these forms, a feature that goes back to the Greek philosopher Plato and his school in the fifth century B.C.

For a theorem the method of analysis consists in reasoning as follows: "I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that; I can prove that if I can prove a third thing"; and so the reasoning runs until the pupil comes to the point where he is able to add, "but I *can* prove that." This does not prove the proposition, but it enables him to reverse the process, beginning with the thing he can prove and going back, step by step, to the thing that he is to prove. Analysis is, therefore, his method of discovery of the way in which he may arrange his synthetic proof. Pupils often wonder how any one ever came to know how to arrange the proofs of geometry, and this answers the question. Someone guessed that a statement was true; he applied analysis and found that he *could* prove it, he then applied synthesis and *did* prove it.

For a problem the method of analysis is much the same as for a theorem. Two things are involved, however, instead of one, for here we must make the construction and then prove that this construction is correct. The pupil, therefore, first supposes the problem solved, and sees what results follow. He then reverses the process and sees if he can attain these results and thus effect the required construction. If so, he states the process and gives the resulting proof. For example:



In a $\triangle ABC$ construct $PQ \parallel$ to the base AB, cutting the sides in P and Q, so that PQ shall equal $AP + BQ$.

Analysis. Assume the problem solved.

Then AP must equal some part of PQ, as PX, and BQ must equal QX.

But if $AP = PX$, $\angle PXA$ must equal $\angle XAP$.

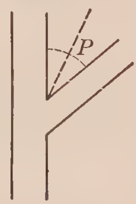
Since PQ is $\parallel$ to AB, $\angle PXA = \angle BAX$.

Then $\angle BAX$ must equal $\angle XAP$.

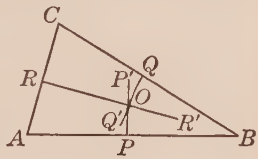
Similarly, $\angle QBX = \angle XBA$. Why?

Construction. Now reverse the process. What may we do to $\measuredangle A$ and B in order to fix X? Then how shall PQ be drawn? Now give the proof.

Method of Loci. The third general method of attack applies chiefly to problems where some point is to be determined. This is the method of the intersection of loci. Thus, to locate an electric light at a point 18 ft. from the point of intersection of two streets and equidistant from them, one locus is evidently a circle with a radius of 18 ft. and with its center at the vertex of the angle made by the streets, and the other locus is the bisector of the angle.

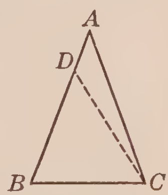


The method is also occasionally applicable to theorems. For example, suppose that we have to prove that the three perpendicular bisectors of the sides of a triangle pass through the same point. We know that the locus of points equidistant from A and B is PP', and the locus of points equidistant from B and C is QQ'. These can easily be shown to intersect, as at O. Then O, being equidistant from A, B, and C, is also on the perpendicular bisector of AC. Therefore these bisectors are concurrent in O.



These are the chief methods of attack, and are all that should be given to an ordinary class for practical use.

Indirect Method. Although the methods already mentioned are the ones that will be used chiefly, there is another that is occasionally needed as a last resort. It consists in supposing that a statement is false and in showing that this supposition leads to conclusions that are impossible. This being the case, we infer that the supposition that the statement is false is itself a false supposition, and that therefore the statement is true. For example, suppose that we wish to prove that if the $\measuredangle$ B and C of the $\triangle ABC$ are equal, the sides AC and AB opposite these angles are equal. We may proceed in the following manner:



We first suppose that AC and AB are not equal, and that, in this figure, $AB > AC$.

Then from AB cut off DB equal to AC.

Then in $\triangle$ DBC and ACB, we have

$DB = AC$, By construction

$BC = BC$, Identical

and $\angle CBD = \angle ACB$, Given

and hence $\triangle DBC$ is congruent to $\triangle ACB$, or a part of $\triangle ACB$ is equal to the whole, which is impossible.

Hence our supposition leads to an absurdity, and it must therefore be false.

This was Euclid's (about 300 B.C.) method of proving this proposition.

We may give special names to this method of proof, as also to some phases of the other methods, but it does not help the student to do so. We may, for example, speak of the indirect method as the method of elimination, all possibilities having been eliminated except the one that the statement to be proved must be true. There have been several books written upon different methods, and it would be an easy thing to make rather an extended list of schemes for attacking propositions. This might interest some who have pleasure in new names for trivial or discarded methods, but it would hurt rather than help the cause of geometry.

Since the inductive and deductive methods of development are discussed in Chapter IX, we shall not take them up at this time.

General Directions. Besides the methods of attack, there are a few general directions that should be given to pupils who are studying geometry, as follows:

1. *Read the proposition carefully.* This means that you should be sure that the pupil has clearly in mind what is given, and that he should keep it there as he works. It will help him if he draws a neat figure freehand while he reads the proposition.

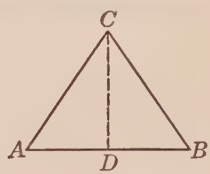
2. *Take a general figure.* This means that the pupil, in attacking either a theorem or a problem, should take the most general figure possible. Thus, if a proposition relates to a quadrilateral, he should take one with unequal sides and unequal angles rather than a square or even a rectangle. The simpler figures often

deceive a pupil into feeling that he has a proof of general applicability when in reality he has one only for a special case.

3. *Draw the figure carefully.* In some cases the pupil will find it helpful actually to construct the figure with straightedge and compasses, since the construction will help him to see the relations leading to a proof. This is the case, for example, in proving that certain lines in a triangle, like the medians, pass through a single point.

4. *Understand clearly what is given and what is required.* This means that the pupil should set forth very exactly the thing that is given, using letters relating to the figure that has been drawn. He should then set forth with the same exactness the thing that is to be proved. The neglect to do this is the cause of a large percentage of the failures. The knowing of exactly what we have to do and exactly what we have with which to do it is half the battle.

5. *Avoid all vagueness of terms.* This means that if a proposition seems hazy, the difficulty is probably with the wording. In this case let the pupil try substituting the definition for the name of the thing defined. Thus instead of thinking too long about proving that the median to the base of an isosceles triangle is perpendicular to the base, let him draw the figure and think that there is given



$$AC = BC,$$
$$AD = BD,$$

and that there is to be proved that

$$\angle CDA = \angle CDB.$$

Thus we replace "median," "isosceles," and "perpendicular" by statements that express the same idea more simply.

6. *Think out a plan for proof.* This means that the pupil should not only state precisely what is given and precisely what is to be proved, but should try to see just what should be added to the given conditions to establish the proof. If, for example, he is trying to prove two lines equal, he should ask himself the question "What are the methods of proving lines equal?" He should then select the one that is best suited to his needs.

8. Extent to which Algebra should be Used

The Early View of the Question. As has been said before, our elementary geometry is an ancient Greek invention, or at least it received its only noteworthy development in ancient times in the schools of Greece and her colonies. The Greek mind was singularly logical and was given to the search after truth for its own sake. Our elementary geometry was developed largely between the years 500 and 300 B.C., during which period the Greeks had no algebra in the sense that we now use the term. Moreover, even if they had known algebra, it is not probable that they would have used it in connection with geometry. Such an attempt would have seemed to their minds very illogical.

If we say that two similar solids (s, s') are to each other as the cubes of any two corresponding lines (x, x'), that is, that

$$\frac{s}{s'} = \frac{x^3}{x'^3},$$

we do not then hesitate to "clear of fractions" and to write

$$sx'^3 = s'x^3,$$

or to manipulate the equation in any other algebraic fashion.

The Greeks, however, would have said, "You can multiply a solid by 2 or by 3, or any other integer, but how can you multiply it by a line? Even if you could do that, how can you multiply one solid by another; for example, a cube by a cube?"

If we replied that we were using numbers to represent geometric magnitudes, and that we can multiply by a number, the Greeks would have said, "Suppose that your multiplier is the number representing the diagonal of a square whose side is 1; it will then be represented by $\sqrt{2}$, and how can you multiply by that? I have 10 drachmas (the name for certain Greek coins), and you have $\sqrt{2}$ times as many, how many do you have? What does it mean to pick up a drachma the square root of two times, or to look at it the cube root of seven times?"

And so the early Greeks banished from pure geometry all work with numbers, and this means that they would not have allowed the use of algebraic letters which represent numbers,

even if they had known anything about them. They therefore worked out a pure geometry, and they and their successors maintained this purity, except in applications to measurement, which were offshoots rather than integral parts of geometry. Even in the treatment of proportion the work was purely geometric, the proposition about the product of the extremes being stated substantially as follows: "If four line segments are in proportion, the rectangle of the means is equal to the rectangle of the extremes," the actual rectangles being drawn. It was not until modern times that any algebra was allowed in the work in geometry.

The Question of Proportion. As already said, the ancient geometers treated proportion by pure geometry. At the present time it is treated algebraically, one reason being that we have a simple algebraic symbolism which was quite unknown before the seventeenth century, and the other being that the pure geometric treatment is so difficult that it cannot be used by the majority of the students in our secondary schools.

It has therefore come to be the custom to consider a proportion as a simple type of algebraic equation, abandoning such terms as "alternation" and "inversion," and such statements as that about the "product (or rectangle) of the means."

Some teachers have favored either the breaking away from geometry at this point and spending some time on algebra or, if they are teaching algebra, the breaking away from that and spending some time on geometry. It is quite as if, in the work in mensuration, when they wished to use multiplication, with which the pupil has long been familiar, they should break away from measurements for the purpose of spending some time on arithmetic. The student has long been familiar with such a simple equation as

$$\frac{x}{b}=\frac{c}{d}$$

when he begins his theory of proportional magnitudes in geometry, and he should spend a lesson period in reviewing it with a view to his immediate needs and then proceed with his geometry.

Algebra used in Proofs. Every part of mathematics with which the pupil is familiar should be used freely in geometry,

except so far as it takes for granted a thing which geometry is supposed to prove. For example, it would not be legitimate to use trigonometry for the purpose of proving the Pythagorean Theorem, because trigonometry is based in part upon this theorem, it being assumed as true when trigonometry is studied in our modern courses in algebra.

It would, however, be proper to prove this theorem algebraically as soon as similar triangles have been studied. The proof has already been given (page 238) and need not be repeated. It is accepted by our best writers on geometry, the purely geometric one often being given later as an optional form. Since the latter shows the actual geometric squares, it has some advantage in that it visualizes the figure.

It may be said, therefore, that algebra should be used wherever it naturally finds a place, but it should not be forced in where it does not fit. The extreme idea that there is a close union between algebra and demonstrative geometry has never had any considerable number of thoughtful advocates, and it is even less talked of at present than it was a few years ago.

9. The Conduct of a Class

Inadvisability of Definite Rules. No definite rules can be given for the detailed conduct of a class in any subject. If it were possible to formulate such rules, all the personal magnetism of the teacher, all the enthusiasm, all the originality, all the spirit of the class, would depart, and we should have a dull, dry mechanism. There is no single best method of teaching geometry or anything else. The experience of the schools has been such as to show that a few great principles stand out as generally accepted, but as to the carrying out of these principles there can be no definite rules.

Employment of Time. Let us first consider the general question of the employment of time in a recitation in geometry. We might all agree on certain general principles, and yet no two teachers ever would or even should divide the period even approximately in the same way. First, a class should have an

opportunity to ask questions. A teacher here shows his power at its best, listening sympathetically to any good question, quickly seeing the essential point, and either answering it or restating it in such a way that the pupil can answer it for himself. Certain questions should be answered by the teacher; he is there for that purpose. Others can at once be put in such a light that the pupil can himself answer them. Others may better be answered by the class. Occasionally, but more rarely, a pupil may be told to "look that up for tomorrow," a plan that is commonly considered by students as a confession of weakness on the part of the teacher, as it probably is. Of course a class will waste time in questioning a weak teacher, but a strong one need have no fear on this account. Five minutes given at the opening of a recitation to brisk, pointed questions by the class, with the same credit given to a good question as to a good answer, will do a great deal to create a spirit of comradeship, of frankness, and of honesty, and will reveal to a sympathetic teacher the difficulties of a class much better than the same amount of time devoted to blackboard work. But there must be no slothful thinking, and the class must feel that it has only a limited time, say five minutes at the most, to get the help it needs.

Division of Time in a Class Period. Next in order may be the teacher's report on any papers that the class has handed in. It is impossible to tell how much of this paper work should be demanded. The local school conditions, the mental condition of the class, and the time at the disposal of the teacher are all factors in the case. In general, it may be said that enough of this kind of work should be required to enable the teacher to know if the pupils are neat and accurate in setting down their demonstrations. On the other hand, paper work gives an opportunity for dishonesty, and it consumes a great deal of the teacher's time that might better be given to reading good books on the subject that he is teaching. If, however, any papers have been submitted, about five minutes may well be given to a rapid review of the failures and the successes. In general, it is good educational policy to speak of the errors and failures

impersonally, but occasionally to mention by name any one who has done a piece of work that is worthy of special comment. Pupils may better be praised in public and have their errors pointed out in private. There is such a thing, however, as praising too much, when nothing worthy of note has been done, just as there is danger of blaming too much, which quickly degenerates into mere "nagging."

Assigning the Advance Lesson. The third division of the recitation period may profitably go to assigning the advance lesson. The class questions and the teacher's report on written work have shown the mental status of the pupils, so that the teacher now knows what he may expect for the next lesson. If he assigns his lesson at the beginning of the period, he does not have this information. If he waits to the end, he may be too hurried to give any "development" that the new lesson may require. There can be no rule as to how to assign a new lesson; it all depends upon what the lesson is, upon the mental state of the class, and not a little upon the idiosyncrasies of the teacher.

The Student's Part in the Program. The fourth division of the period should be reached, in general, in about ten minutes. This includes the so-called recitation proper. But as to the nature of this work no very definite rules can be laid down, and this is very fortunate; otherwise all our personality would be lost in a mere mechanical process. A few suggestions as to how the time may be spent profitably will, however, be given.

In general, a pupil should state the theorem quickly, state exactly what is given and what is to be proved with respect to the figure, and then give the proof. At first it is desirable that he should give the reason for each step in full, and later give the essential part in a few words. It is better to avoid the expression "by previous proposition," for it soon comes to be abused, and of course the learning of section numbers in a book is a barbarism. It is only by continually stating the propositions used that a pupil comes to have well fixed in his memory the basal theorems of geometry, and without these he cannot make progress in his subsequent mathematics. In general, it is better to allow a pupil to finish his proof before asking him

any questions, the constant interruptions indulged in by some teachers being the cause of no little confusion and hesitancy on the part of pupils. Sometimes it is well to have a figure drawn differently from the one in the book, or lettered differently, so as to make sure that the pupil has not memorized the proof; but in general such devices are unnecessary, for a teacher can easily discover whether the proof is thoroughly understood, either by the manner of the pupil or by some slight questioning. A good textbook has the figures systematically lettered in some helpful way that is easily followed by the class that is listening to the recitation, and it is not advisable to abandon this for a random set of letters arranged in no proper order.

It is good educational policy for the teacher to commend at least as often as he finds fault when criticizing a recitation at the blackboard and when discussing the pupils' papers. Optimism, encouragement, sympathy, the genuine desire to help, the putting of one's self in the pupil's place, the doing to the pupil as the teacher would that he should do in return, — these are educational policies that make for better geometry as they make for better life.

The prime failure in teaching geometry lies unquestionably in the lack of interest on the part of the pupil, and this has been brought about by the ancient plan of simply reading and memorizing proofs. It is to get away from this that teachers resort to some such development of the lesson in advance as has been suggested above. It is usually a good plan to give the easier propositions as exercises before they are reached in the text, if this can be conveniently done. An English writer has recently contributed this further idea:

It might be more stimulating to encourage investigation than to demand proofs of stated facts; that is, to say, "Here is a figure drawn in this way; find out anything you can about it." Some such exercises having been performed jointly by teachers and pupils, the lust of investigation and healthy competition which is present in every normal boy or girl might be awakened so far as to make such little researches really attractive; moreover, the training thus given is of far more value than that obtained by proving facts which are stated in advance, for it is seldom, if ever, that the problems of adult life present

themselves in this manner. The spirit of the question "What is true?" is positive and constructive, but that involved in "Is this true?" is negative and destructive.[1]

Home Assignment of Exercises. It is good teaching practice to assign four or five exercises each day for home work. One of these exercises may then be assigned the first thing the next day for written work in class. This limitation saves the pupil much time, and it also saves time for the teacher, there being only one exercise to examine. In the long run this plan is an excellent way of testing a pupil's geometric powers.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Why should the course in mathematics discussed in the previous pages be required of every pupil in the junior high school?

2. Be able to argue for or against the making of all mathematics beyond the junior high school elective.

3. Review whatever argument you consider sound for permitting every boy and girl to study a short unit of demonstrative geometry in the junior high school.

4. State and show how you would prove a one-step and a two-step exercise, each based on the second congruence theorem, — *Two triangles are congruent if two angles and the included side of one are respectively equal to two angles and the included side of the other.*

5. Show that the theorem which says, *Two triangles are congruent if three sides of the one are respectively equal to three sides of the other,* is a corollary of the theorem referred to on page 236.

6. State and show how you would prove a one-step and a two-step exercise, each based on the congruence theorem stated in Ex. 5.

7. Outline a list of propositions that you would use in teaching a six-weeks' course in demonstrative geometry, indicating which ones you would prove and which you would take for granted.

8. Give an example of a proposition in geometry not given in this book where algebra could be used in the proof.

9. Select a list of five original exercises that you consider good practical applications of the propositions you gave in No. 6.

[1] G. St. L. Carson, Essays on Mathematical Education.

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CHAPTER IX

SUPERVISION AND INSTRUCTION IN MATHEMATICS

1. Lesson Types and Methods of Teaching

Purpose of the Chapter. It is the purpose of this chapter to set forth some of the problems that must be met by both the classroom teacher of mathematics and the supervisor. It is not desirable to attempt to discuss separately the needs and duties of each. The successful supervisor is not a faultfinder; he is a helper, and in order to assist the teacher he must know what good teaching means. It follows that the suggestions that may appropriately be made to either will be of distinct benefit to the other.

Lesson Types. Although it is impossible in the space at our disposal to mention all lesson types that various writers have suggested, most of these types being artificial and of no practical merit, the basic principles of a few of the most important ones will be set forth and their significance explained. Those who wish for a more detailed discussion of the subject, or of the question of supervision in general, should consult such works as deal directly with this phase of education.

Classification of Lesson Types. In this presentation of the subject we shall confine ourselves to seven types, as follows:

1. *Inductive development,* by which is meant such a development as proceeds from a few special cases to the inferring of a general law.

For example, after a child has seen that $2 \times 3 = 3 \times 2$, $3 \times 1 = 1 \times 3$, and $4 \times 2 = 2 \times 4$, he will easily infer the general law that we obtain the same result if we interchange the two numbers in multiplication.

2. *Drill lesson,* by which we mean a lesson concerned chiefly with fixing permanently in mind a certain set of facts.

For example, we drill upon such "troublesome groups" as $8 + 7$, $5 + 8$, $9 + 7$, and $7 + 5$; upon such word lists as *exponent, factor, coefficient,* and *binomial*; or upon such laws as $(a^m)^n = a^{mn}$.

3. *Deductive development*, by which we mean that the inductive development has led to a general understanding that $a^2a^3 = a^5$; that the drill lesson has fixed this general law in mind; and that the time has now arrived for deducing a statement of this law.

For example, (1) $a^2a^3 = a^5$, "I see that I should add the exponents" (induction); (2) $a^2a^4 = a^6$, $a^3a^3 = a^6$, $a^3a^4 = a^7$ (drill); (3) "the general law is expressed easily as $a^m a^n = a^{m+n}$" (deduction).

4. *Review lesson*, by which we mean that we should seek to accomplish two major purposes: (1) we should make certain that the pupils are keeping in mind the basal features already acquired; and (2) we should lead them to look, as from a hilltop, over the route which has been traveled, thus enabling them to obtain a much broader view than they had when walking along the road, — a road often dusty and sometimes stony. In the language of educators, we seek to give the pupil an understanding of the logical organization of the material which he has been studying.

For example, when a student is working upon the geometry of triangles and parallels, he sees each theorem as a thing in itself. When he reviews the chapter, he sees these theorems as links in a strong chain of reasoning; in other words, he organizes his material.

5. *Examination lesson*, by which we mean a period set aside for a written examination upon work accomplished. Such a lesson should reveal the value of concentration upon the problem of extracting from a mass of acquired material the essentials that bear upon the topics called for in the examination.

For example, if the pupil is asked to state, for each of the three most important congruence theorems, a single basal proposition depending directly upon it, he must concentrate his mind upon the problem of extracting the information from a considerable mass of material. He will also reveal his knowledge as to the great question of dependence of one proposition upon another.

6. *Appreciation lesson*, by which we mean a lesson designed especially to develop pleasure in the cultivation of some particular field of work, to develop genuine delight in the beauties of a subject, whether the subject be painting or music, science or mathematics.

For example, a pupil who studies a regular hexagon without seeing its relation to a quartz crystal or a snowdrop, or a pentagon without connecting it with the cross section of an apple or of the seed pod of a rose, has missed much in his appreciation of geometry and of the meaning of the old Greek phrase "God eternally geometrizes."

7. *A conversation lesson*, by which we mean what is also called a "lecture lesson," — a lesson in which the teacher teaches, talks to the class, tells things which the textbook has not the space to tell, and opens up a vision of future work that is so dependent upon the needs of the individual class as to be outside the range of possibility of any book. The term "lecture lesson" is unfortunate; it suggests college lectures, and these have the reputation of being frequently so uninteresting as to be depressing. The name, however, is not so important as the thing itself, and this has, at its best, a very great value. There are so many interesting things about mathematics that a textbook cannot tell, so many inspiring things that a class is hungry to absorb, and so many beautiful things that tend to change a pupil's whole attitude of mind toward mathematics that these conversation lessons should play a much greater part in teaching than they now do.

For example, imagine an animated, gifted, interesting teacher, sitting in the midst of a class of pupils in, say, Grade VI, and telling the story of the numerals that the world has used: of how they were at one time written on clay tablets, later upon papyrus, still later on metal, or parchment, or stone, or wood; and of the coming of paper, of quill pens, of metal pens, of printing, and of typewriting. If it should lead up to the Chinese laundryman's abacus, to the cash register, and to the modern calculating machine, how could any class fail to be intensely interested in the story? What is required of the teacher to make it a success? The reading of good books upon the subject taught, an interest in the subject itself, and the ability to speak with honesty of purpose, with earnestness, and with the desire to help the class and to make the work attractive.

Danger of Overlapping. It will naturally be asked whether any one of these lesson types must not necessarily infringe upon the domain of another. The question almost answers itself. Of course there will be constant and desirable infringement. The inductive development may lead to drill in the same lesson, but it is certain to be followed by much drill in subsequent ones. The drill will possibly lead to deduction in the same lesson, but

the teacher is more likely to emphasize this in a subsequent class period. There may be review work in any lesson, but it is probable that the teacher will set aside periods for well-planned reviews of various units of work. There may be an examination during some part of another type of lesson, but this is not usually the case, except with certain modern tests for which only a short time is needed. There is apt to be appreciative work and conversational or lecture features in nearly every lesson, but the teacher will do well to plan such features as the major portions of special lessons arranged for the purpose.

The danger of overlapping is, therefore, not serious. What is of vital importance is that the teacher shall feel that these different types exist, that they have value, and that their essential features should enter into the work of all of us who seek to make that work both interesting and valuable and to arrange it with a sensible view of a systematic presentation of the subject.

Danger of Mechanism. There is, however, a genuine danger of making the work too mechanical. The German educator Herbart, many years ago, laid down "five formal steps" in giving a lesson, and the system was much exploited in this country; but in spite of its value as a suggestion to rather immature teachers in preparing their lesson plans, it never met the expectations of its advocates. The reason for this failure was that it tended to make machines of its followers, running all teachers as well as all lessons into the same mold.

When the question "How shall I teach?" or "What is the method?" is asked, there is no answer such as the questioner expects. A Japanese writer, Motowori, a great authority upon the Shinto faith of his people, once wrote these words: "To have learned that there is no way to be learned and practiced is really to have learned the way of the gods."

The Formal Steps of Herbart. The danger of mechanism may be clearly seen if we state the formal steps of Herbart, doing this in more modern language than that used by his immediate successors and followers. They may be set forth as follows:

1. *Preparation.* In any lesson there should be a brief preparation of the minds of the pupils for the presentation of the new

work. This may include a statement of the nature and purpose of the lesson, so that the pupils may be, even unconsciously, looking ahead and anticipating the teacher's lead. In any case, such preparation should be based upon work already accomplished.

For example, if the subject of the lesson is to be the presentation of directed numbers, the teacher can say that there are some numbers that have a meaning in some cases but not in others. For example, $\frac{3}{4}$ has a meaning when applied to an apple, but none when applied to the number of times you can look out of a window. Similarly, 3° below zero on the thermometer means something definite, but 3° below the center of the earth means nothing. It may then be explained that the class is now ready to consider certain new numbers that in some cases have meaning while in other cases they do not.

Thus the class is brought into a receptive mood and its curiosity is at the same time aroused.

2. *Presentation of the new.* After the proper kind of preparation the pupils are ready for the presentation of the new material. This should be done in as simple language as possible and should follow naturally upon the work of preparation.

For example, if the thermometer has been mentioned and the pupils have seen the marks at zero and below, they are ready to be told that degrees below zero are indicated by the minus sign before the number. There is nothing difficult or particularly strange about the fact; we simply have a brief way of writing "below zero." In other words, a negative number is simply a number below zero on the thermometer, or in playing a game, or in stating financial standing. Similarly, a positive number is above zero.

3. *Establishing bonds.* This means that we seek to tie up our new information with information that we already possess. Herbart spoke of this as "apperception." Fashion changes very frequently in educational language; we now speak of establishing bonds between the new information and the old.

For example, in the case of the negative number we immediately proceed to tie up $-3°$ on the thermometer with a temperature graph; to tie this up with a graph of a man's bank account that has been overdrawn; this with a score in a game, where some scores are below zero; all this with positive and negative weight, with latitude, and with street numbers east and west of a central line.

4. *Generalization.* This means that we pass from simple concrete cases to more abstract and general ones. We draw a line to represent an algebraic scale, the positive numbers at the

right of a zero point and the negative ones at the left. Upon this scale we show fractions and surds as well as integral numbers, and we then come to speak of -3 and $+3$ just as readily as we do of $-3°$ or \$3, and of $-\frac{3}{4}$ as familiarly as of -2.

This will probably not be done in the same lesson as the first three steps, nor will the generalization offered to most beginning classes extend to such topics as the multiplication and division of these numbers. It will, however, extend to their addition and subtraction.

5. *Application*. Such a subject as directed numbers, for example, has little meaning unless it is applied. Indeed, elementary arithmetic and algebra signify little to any pupil unless he sees how they are to be used. If well taught, he will appreciate their beauty, their power to fit into his scheme of thought, and their general interest; but their real significance will seem to him to lie in their concrete applications.

For example, the negative number has in itself considerable interest; it is new to the pupil and it fits into the general harmony of a number system. What makes a pupil wish to study it, however, is largely his perception of the fact that it can be and is so generally used. This use is shown in any progressive modern algebra much more effectively than was formerly done.

The Failure of Herbart. The formal steps of Herbart had a sound psychological basis. They represent the steps that any gifted teacher naturally takes in teaching something new. They are valuable in their suggestiveness to an inexperienced teacher. In spite of all this, however, they failed to meet the expectations of their advocates, — not because they were steps, but because they were formal. A gifted teacher followed their general sequence instinctively; an inexperienced teacher, particularly one not favored with a clear vision of objectives, failed because he tended to think of the steps rather than of the child's interests, his life, his surroundings, and his human needs. He said to himself, "I must now change to generalization," — the scheme overshadowed the interests of the child, and the teacher became wooden in his methods.

The Danger of Lesson Types. The danger encountered by Herbart's formal steps faces us when we consider the lesson types mentioned. If we feel that we must have inductive development on Monday, a drill lesson on Tuesday, a deductive

development the next day, and so on through the week, our mental state is quite hopeless. If we feel that we must necessarily give a full class period to any one of these types, it is equally so. But if we see that these types are psychologically valuable and that we should use common sense in employing them, then they will help all of us, experienced and inexperienced alike.

2. The Lesson Types in Detail

Inductive Development Lesson. Mathematics in the making is inductive. Deduction is a late development, both in the history of the race and in the history of the individual. It is therefore entirely natural that any topic of elementary mathematics should begin with induction, — with the consideration of the special case. This has always characterized the work of the successful teacher and of the successful discoverer of great mathematical truths as well. The teacher who would begin the teaching of the Binomial Theorem by attempting to multiply $a + b$ by itself $n - 1$ times would, of course, fail; but the one who begins with the first three powers of $a + b$, starting with $(a + b)^2$, and inductively derives the rule from these results has prepared the way for the intelligent deduction of the general law. Although this treatment involves deduction of a simple nature as its final step, in the student's mind there is a definite line of cleavage between the two methods. He comes to feel the value of induction in attacking new problems and, indeed, to realize its superiority for this purpose. Discoveries in mathematics, as in life generally, are made by induction; the formal proofs of the results may be made by deduction.

Steps in an Inductive Lesson. The first step in an inductive lesson presents two problems:

1. To find a basis in former experience upon which to build.
2. To inculcate a feeling of need for the new fact about to be presented.

These two problems cover most of Herbart's first formal step as stated on page 273.

The second step seeks to focus the pupil's mind upon the

new topic, to unite the new material with the old, and to lead to the formation of new judgments as to the facts presented. This is essentially Herbart's steps of establishing bonds and of generalization.

The third step is deductive in its nature, although the general trend of the lesson is inductive. It carries the pupil far enough to allow him to see that a general law is involved, and to apply this law, even before he can state it succinctly, to special cases.

For example, after seeing that $2^2 \times 2^3 = 4 \times 8 = 32$, and that $2^5 = 32$, it is apparent that $2^2 \times 2^3 = 32$, or $2^2 \times 2^3 = 2^5$. If the pupil then sees that $3^2 \times 3^3 = 3^5$, he can probably see that $2^3 \times 2^4 = 2^7$, without any deductive proof or any formal statement of the law.

Drill Lesson. The objective in the drill lesson is to make pupils' responses automatic. The technique of drill is therefore based upon the ideas of habit formation and of concentration of attention. Given these, success can be attained; without them there will be failure. The conduct of the lesson will be systematized by keeping in mind the following points:

1. The work should be properly motivated; that is, the pupil must feel a need for the drill and must thus be conscious of a driving purpose.

2. There must be a careful assignment of the work, so that he will know precisely what he is expected to do and the time which is at his disposal.

3. Interested attention must accompany repetition. Psychology has modified the old adage that "practice makes perfect," so that it reads "interested practice makes perfect."

4. The attention of the pupils can be secured by such devices as a variation in procedure, the setting of time limits, and an appeal to emulation.

For example, if a teacher plans to allow 15 minutes on the law of signs in algebraic subtraction, part of the time may be taken for oral work and part for written. In setting time limits it is better to ask the pupils to see how many examples they can do in 10 minutes rather than to work for 10 minutes without this stimulus.

With respect to emulation, it is better to have a pupil work to excel his own previous record than to excel the record of someone else, except in the case of very evenly matched individuals.

5. In drill work we seek for absolute accuracy; in measurement we seek for approximate accuracy.

That is, we measure to the nearest unit of some kind, — say to the nearest eighth of an inch.

6. In drilling upon a series of related facts, every member of the series should be included.

For example, in the case of the law of signs in algebraic addition every combination of signs should enter into the drill.

7. There should be concentrated practice on a few skills rather than a large amount on each of a great many.

8. The most effective kind of drill is of an individual nature.

9. Drill exercises should be accompanied by a method of scoring so that the pupil may measure his own success.

10. Drill should be distributed in decreasing amounts and at increasing intervals.

11. Drill should be on specific skills.

12. Drill periods should be relatively short, depending somewhat upon the grade in which the work is done.

13. A drill exercise should be standardized, in order that a pupil may have some idea of the skill he has attained.

14. Items presenting the greatest difficulty should have more attention than others.

For example, there should be more drill upon the case of $a - (b - c)$ than upon that of $a + (b - c)$.

15. So far as possible, the teacher should plan the drill work for that part of the day in which the pupils' minds are most active.

16. The teacher should remember that the brighter pupils deserve at least as much attention as the duller ones. He should give them as much work to do as their abilities demand. This can be done by selecting more facts from the troublesome number combinations in arithmetic or from the harder cases in algebra and geometry. The world suffers more by retarding the possible genius than by retarding the dullard, if only we are fairly sure which is which. In general, the one with a high intelligence quotient will do better work than the one with a low intelligence quotient. The latter may take more time, but is not entitled to monopolize the teacher's attention.

17. In written drill, each pupil should have an exercise book in which his work is kept. It enables him to keep track of his progress and allows his parents to add their encouragement to that which the teacher may properly give. While the work is being done the teacher will find it desirable to pass around the class quickly, encouraging the weaker pupils and assisting the stronger ones to progress as far and as rapidly as their abilities justify.

Deductive Development Lesson. Though somewhat overlapping the inductive treatment, this type of development lesson has its distinctive features. It seeks to deduce general laws from our previous induction (see page 271). For this purpose it makes use of principles, definitions, and conclusions already studied. It then applies these general laws to problem solving, — the application stage of the Herbart scheme. The general nature of such a lesson may be understood by considering the following steps and the accompanying illustration:

1. The problem should be distinctly stated. For example, it may be to solve a set of quadratic equations.

2. The student should consider the different methods he has already mastered, — say the graph, factoring, completing the square, and the formula.

3. He should look at the equations and decide upon the method that will most readily apply at least to the majority of the equations in the set, stating the reasons for the choice.

4. He should then deduce a rule for choosing the best method to be used for solving certain types of quadratics.

For example, he should see that the equation $x^2 - 7x + 12 = 0$ can be solved by a graph, but that it takes too long to construct it, and in any case the solution may be merely approximate; that it can be solved by completing the square or by the formula, but that either of these plans is too long; and that it is evidently a made-up case in which the factors, $x - 3$ and $x - 4$, are readily seen, so that, at sight, the roots are apparently 3 and 4.

Review Lesson. The review lesson should summarize the important features of the preceding work. It should arrange for a systematic repetition of facts, adding to the interest by varying the methods of treatment. No review is worth giving if it merely repeats facts and principles in exactly the same way as they were originally given.

The most successful review is that which approaches the subject from a new angle. It differs from a drill lesson by emphasizing thought processes rather than habit formation and mere automatic responses.

We should remember, however, that in a review lesson we should not merely repeat, or have the pupils repeat, the facts of the previous lesson even if complete mastery has not been attained. As intimated before, to do this is to dull the interest and lessen the chance of ultimate mastery. We must have the pupil see the facts in a new light, and appreciate relationships and significant points that did not stand out the first time he went over the material.

Examination Lesson. There has for many years been a protest against the older type of examination. This is based chiefly upon four beliefs: (1) the examinations tend to perpetuate the use of matter that has lost whatever value it may once have had, which at best was very slight; (2) they lead to a so-called "cramming" process which tends merely to temporary efficiency; (3) they induce a nervous strain that is not justified by the results; and (4) they are often not so efficient as the more modern new-type test.

As to the first of these statements there can be no doubt. These examinations are responsible for the persistence of types of work in the division of polynomials, in fractions that are never used, in factoring that has no applications to genuine cases, in surds, and in certain types of equations that are unwarranted and that should be discarded. Among the worst offenders are examinations set by certain state and city authorities.

The second objection is not so serious as the first, but it is real and such practices should be discouraged.

The third objection arises largely from the fact that we now have a large proportion of girls in our schools, which was not formerly the case. For physiological reasons it is desirable to treat them differently from boys, and in mixed schools there is some fair excuse for objecting to long and difficult examinations. If, however, the girl is to be held to the same intellectual standard as the boy, some method of measurement must be

found. It is probable that this will be done through the modern combination of examination and school records.

The fourth objection remains to be proved. It seems to be valid in certain subjects involving mere information relating to simple facts, but as yet there seems to be no satisfactory brief test of a student's ability to concentrate for a considerable period upon a geometric proof of any difficulty, or upon a problem involving a considerable number of possible avenues of approach.

It therefore seems that the schools will do better to accustom their students to written examinations of one class period than to assume that such tests of ability have outlived their usefulness. After all, if a student knows a subject, should he not be ready to prove that he knows it?

Nature of an Examination Lesson. The following suggestions relate to the nature and the fundamental values of an examination lesson:

1. Its formal nature is of itself valuable.
2. The fact that it assists in organizing a pupil's knowledge is important.
3. It has value in that it secures a high degree of attention immediately before and while it is given.
4. It should include those elements of the subject tested which are of the greatest permanent value.
5. A final examination should test (1) the content of the course, (2) the ability to apply and use the subject, and (3) the higher ideals developed, — ethical, social, and those that include the play of imagination.

Appreciation Lesson. Everyone who is familiar with the problems of education will recognize the necessity of cultivating an appreciation of good books, of beautiful pictures, of fine music, of well-planned cities and parks, of artistic homes, and of the beauties of the natural and mathematical sciences. This being the case, the schools should provide definite training to secure the necessary results.

Many and perhaps the majority of teachers feel that their duty lies merely in imparting the knowledge of certain facts as

required by some particular course of study, but they are not the leaders in their profession. A real teacher always seeks to arouse a love for the subject taught and an appreciation of its significance in the lives of the people. To him the subject taught, while valuable in itself, is also a means for cultivating higher ideals in the student's life.

An illustration of the failure to "reach Central" is seen in the actual case of an uninspiring teacher who had been conducting a lesson in which the story of Arnold von Winkelried was being read. Thinking that the recital itself had aroused the necessary emotions of patriotism she asked one of the brightest pupils how the story impressed him, receiving the weary reply: "Well, I feel as if I never want to read it again." The idea of a real appreciation lesson had never made any impression upon that teacher.

Nature of the Appreciation Lesson. The appreciation felt by a pupil tends to be either social or æsthetic. Under the former heading are naturally included such phases as the ethical and religious, and under the latter such feelings as are suggested by the ideas of grandeur, the infinite, and the harmonious. The teacher who is most successful in developing an appreciation of the significance of the subject taught is one who has himself had a generous training in the same line and a rich experience in searching for the finer elements of life.

In addition to such training the teacher's greatest assets are (1) a knowledge of the subject taught, — not merely of the textbook used; (2) a love for the beautiful, — not merely in an art gallery but in the gallery of humanity; and (3) a habit of giving play to the imagination, — not merely in a book of poems but in a book on geometry.

Preparing for the Appreciation Lesson. While the teacher will do well to read, first, pages 27–32 of this book in order to feel the general spirit of appreciation of the finer elements of mathematics, the following questions may be of some assistance in preparing for a lesson of this type:

1. Has the teacher a firm belief that the subject should be taught, — that it is of definite value in the life of every student

in the class, and that, aside from its utility, it is possessed of inherent beauty and nobility? If so, this belief is almost certain to find its way unconsciously to the minds of those whom he teaches.

2. Does the teacher understand the purpose of the author of the textbook in use? Has he read with care the preface and has he studied the general plan as set forth in the table of contents? If so, does he feel in sympathy with the author and can he assist in carrying out his plan? What is more important still, can he use the book as a structure upon which to build up for his pupils an appreciation of the commercial value and the spiritual value of the subject itself?

3. Has the teacher a sufficient background of appreciation of the subject, — of its influence upon the world and its real significance in modern life?

4. Does the teacher think of the subject only as it was taught to him, — perhaps very stupidly? If his pupils think it dull, where does the difficulty lie, — with them or with him?

5. If the answers to these questions show that he lacks the necessary enthusiasm, can he, by sincere self-examination and by a fresh approach to the subject, awaken within himself a new impulse to make mathematics abundantly worth while to those entrusted to his care? If not, then mathematics is not the subject for him, and the sooner he takes up some other branch of education the sooner he will be likely to find his special road to success.

Conversation Lesson. There are various other special types of lesson besides those referred to on pages 270–272, such as the "Class and Board" lesson and the "Speak and Write" recitation; but most of these are included under the general heading of Conversation Lesson. This comprises the lecture, the informal discussion of any silent reading lessons that may be given, and other forms of instruction in which the teacher actually teaches.

The preparation stage of any lesson requires a forward look for the purpose of establishing a motive for further study. It also requires a retrospective view to show the pupil how far he

has progressed and what landmarks he has passed. The best way of opening each of these mental views is by an interesting, rather informal talk with the pupils, this being much more effective than a mere lecture to the class.

Another advantage lies in the fact that a teacher is aware of the special needs and interests of the pupils each day, which no textbook writer can foresee, these needs and interests and pupils varying with the locality, the season, and the class itself. This being the case, the teacher is the only person who is so situated as to be able, effectively, to meet the immediate needs of the pupils, and this is best done by a natural, rather informal talk with the class.

The conversation lesson may be thought of as including also the recitation and study lessons. In the former the pupil reports upon such facts as he has learned, and this is most effective when it takes the form of a friendly conversation. It then allows the pupil to state his difficulties freely and affords the teacher an opportunity to clear them away. It also permits of new illustrations and possible applications of the work under consideration, and allows a freedom of question and answer that is very effective. Why should the teacher do all the questioning and the pupil do all the answering? If there is companionship between pupil and teacher, why should the former not be free to ask questions of the teacher, receiving at least as much credit for a well-thought-out question as for an answer that he has learned from a book?

The kinds of questions asked by the teacher are considered at length on pages 291–296.

The Study Lesson. The questions of the study lesson and of supervised study under the teacher's guidance do not, strictly speaking, belong in the discussion of lesson types as we ordinarily interpret the word "lesson." They are therefore referred to at this time rather as an appendix to the discussion than as an integral part of the chapter.

A study lesson, whether supervised in school or intended for home work, is ordinarily an assignment for a subsequent recitation; but it is quite as appropriately given for the purpose of

clarifying a preceding development of some topic or of expressing independent judgment of an assigned book or chapter. In any case the following suggestions should prove of value to the teacher:

1. Study is the independent acquiring of valuable knowledge or of the material for forming worth-while judgments.

For example, we may study the form $a = b \tan A$ for the purpose of memorizing it so that we may use it, or we may study a statistical graph for the purpose of judging the probability as to the increase in value of certain property.

2. It involves sifting out the essentials from a mass of information, and the writing or memorizing of these essentials for future use.

For example, the explanation of the method of multiplication by logarithms involves two essential facts: (1) a logarithm is an exponent, and (2) $a^x a^y = a^{x+y}$. A pupil who sifts out these facts and understands their significance has the whole idea.

3. Pupils differ greatly in ability, — individually, not only as classes or as groups in classes. Supervised study should recognize this fact and should encourage each individual to do his best without being discouraged because he cannot excel his neighbor. The plan of supervision enables the teacher to lessen the amount assigned to the slow student and to increase that assigned to the leaders. As already stated, each should play the game with himself, always endeavoring to increase his own standing without paying attention to outdistancing any except those of about the same ability. Contests are disheartening except with those of the same mental power; handicaps should exist in algebra classes as well as in golf and tennis. Supervised study should give encouragement to the pupil who wishes to master a certain amount of mathematics even if he is backward in his work; he is not responsible for his ancestors and we cannot all be leaders.

Technique of the Study Lesson. Suppose, for example, that the class has a page of original exercises after the third congruence proposition in geometry. It will probably be desirable to recall the fact that in the first place there should be written down precisely what is given and precisely what is to be proved.

The teacher should walk about the class and encourage the slow ones, telling those who have the work right to go ahead with the proof, and finding out where the difficulty lies with the others. He should constantly seek to encourage by approval rather than to discourage through faultfinding.

After the class is well at work he may find it better to take a seat and let anyone who has difficulty come to him. The assistance can best be rendered by a simple question, one that will lead the pupil to think out the next step for himself. If this is not sufficient, the teacher will probably find it desirable to say, "If I were doing this I think I should see if I could bring it either under the first or the third congruence theorem; what do you need in order to bring it under either?" The best plan always is to put the pupil on his own responsibility so far as possible, but not to leave him in a position from which he has not the ability to extricate himself.

Toward the close of the period it is well to walk about the class, asking each how far he has gone and ascertaining where the difficulties, if any, lie. This gives the opportunity for encouraging each one who has done his best, even though others may have done more.

Study Helps. Since the habits of study and the taste for the good and the true are of greater importance than the mere facts of knowledge, the pupil should be assisted in learning how to study, how to use his time economically, and how to make such notes as may be worth while. To this end, the following suggestions should be made as occasion offers:

1. Make a definite daily program, allowing a definite time for each study. This leads to the habit of systematizing your work and of concentrating your thoughts on each subject when its time arrives. No such program will ever be absolutely fixed, but attempt to follow it, shortening some of the periods when possible.

2. Have the necessary material at hand, such as ruler, compasses, protractor, pencil, and paper.

3. Write in your notebook the precise lesson assignment. If the teacher has called attention to any special points of interest

or difficulties to be surmounted, note these briefly and read your notes before you begin to study.

4. Make your textbook your friendly helper, not your master. Do not try to commit it to memory; get the meaning, not the precise wording. The only important exception lies in the case of a very few definitions in geometry that are necessary in later proofs.

5. Lose no time in getting ready for study. Begin work as soon as you sit down. Concentrate at once upon the lesson, holding your mind directly upon the work, letting no outside thoughts disturb you. Cultivate the will to learn.

6. It is frequently a good plan to go over the lesson quickly, getting the general purpose. It may be that this will be sufficient, but usually there are various points that are not entirely clear, and in this case take these up and master them in the order in which they come.

7. In general you will do better to study alone than in a group. Groups waste time in talk. Mathematics is an individual study. After you have studied a lesson, talk it over with others as much as you please, helping them and being helped by getting their points of view.

8. Try to find practical applications for your mathematics in your daily life and surroundings. Always remember, however, that many of the most noble and beautiful things in the world and in subjects like music, poetry, biography, art, and mathematics have no applications to plastering, steel structures, or commerce.

9. Make your study interesting; talk about the interesting things with your parents; try to be as interested in algebra as in music, — they are much alike in many respects; each has rhythm, each has precision, each is truth.

10. Prepare each lesson each day. Do not cultivate the habits of slovenliness and of debt by letting the work pile up ahead of you.

11. If you feel the need of reviews, if you feel the work slipping away from you, tell your teacher. Reviews help you to recall work already done, and they help you to master the subject.

3. Lesson Plans

Importance of Lesson Plans. It is unfortunate that so many beginning teachers fail to realize the importance of a lesson plan. No business man would think of starting a new enterprise without a carefully developed plan of attack; no lawyer would enter upon the trial of a case without giving much thought to his plan of conducting it; no clergyman would preach a sermon without a mathematically arranged sequence of presentation; and no political party would enter upon a campaign without a plan that had been submitted to the critical scrutiny of dozens of experts. But despite all this, a teacher with little or no experience will often enter a class not only with no plan for the lesson but with little thought as to what ground is to be covered, — and then he wonders that his pupils develop no taste for mathematics and no skill in solving its simplest problems.

Even teachers of experience do not hesitate to spend much time and thought upon lessons that have long been familiar to them, changing here and there and adding increased interest through their richer experience and their extended study of the subject to be taught.

For the beginning teacher, however, the value is particularly great for such reasons as the following:

1. It prevents him from wandering too far from the topic considered, keeping the points of the lesson before him in well-planned sequence.

2. It is of great value during the emotional excitement often caused by the presence of visitors, cases of discipline, or the development of side issues due to irrelevant questions asked by the class.

4. Through its careful organization of the material it saves a great deal of time in a year.

5. It aids the teacher in the formation for himself of the habit of conciseness and definiteness, — a habit that will be valuable all through life, outside as well as inside the schoolroom.

6. With reasonable care, a series of lesson plans worked out for one year may form a safe basis for the plans of succeeding

years. By adding new material and ideas, and by omitting features that have proved of little value, the teacher will escape the danger of falling into a rut, while saving himself a great deal of time that would be lost in starting each year entirely afresh.

Form of a Lesson Plan. While the form of a lesson plan is a matter largely of the teacher's own personality, it is interesting to see that one of the best norms we have in mathematics is itself mathematical. In other words, a lesson is not unlike a proof in geometry, where we have (1) a general statement of something to be accomplished (in geometry, probably a theorem to be proved); (2) certain things given with which to work; (3) a statement of certain specific facts to be proved; (4) proof, supported by facts already proved; (5) discussion of the work from different points of view.

In a lesson plan we should have a similarly definite arrangement of work, and we should have it lead similarly to a definite result. The outline will not be the same, but the definiteness of presentation and the succinctness of statement will closely resemble a geometric proof, and like this it will form a basis for discussion and for the play of ingenuity and imagination on the part of the class.

Certain Common Requirements. The following are typical requirements for a workable lesson plan:

1. The basis should be the previous experiences and the present knowledge of the pupils; that is, it should not be what the teacher would like to teach, but what the pupils are prepared to receive.

2. The purpose of teaching the lesson should be made clear at the beginning; that is, the pupils should not be led in the dark, — they should be looking toward the goal.

3. The purpose of studying the lesson should also be made clear, and this is best done by means of a problem of some kind.

The teacher's purpose is, of course, much more far-reaching than the pupils'. Their purpose is to solve some immediate problem, to do a simple piece of work. The teacher's purpose is to forge a connected chain of knowledge of which the immediate problem is merely a link. This purpose cannot often be made clear to the pupils, but the purpose of the day's work can be set forth.

4. The work of the day should be presented with the most interesting illustrations available. Local interests will often suggest material that differs from that given in the textbook. If the teacher can improve upon the textbook he should by all means do so, but this does not mean that time should be wasted in a mere attempt to change what has been well thought out by an expert.

5. The essence of any good plan is the careful organization of the material. In fact, the words "plan" and "organization" are practically synonymous. The work should be arranged about a few points of substantially equal importance.

6. The plan should include a few clear questions, each such as to encourage independent thought and originality of expression on the part of the pupil. He is a thinking being, not a machine built to reproduce words from a textbook.

7. Just so far as the abilities of the class permit, the lesson plan should reveal the development of a topic, not a topic already developed. A pupil in a class in nature study takes much more interest in watching a plant develop from day to day, living its humble life, than in simply looking at a plant that has already developed. In the same way, a student in the first stages of demonstrative geometry takes far more interest in seeing a proof develop, and in taking part in the work, than he can possibly take in reading a proof that is formally expressed in a textbook.

8. The plan should arrange for the major expenditure of time and effort upon the more important parts of the work in hand. Inexperienced teachers are apt to allow a great deal of time to be wasted upon the relatively unimportant parts of a lesson, and a well-considered lesson plan will be found very helpful in overcoming this difficulty.

9. The principles brought out in the lesson should be applied to a wider field whenever this is feasible. For example, certain conditions and arguments in geometry apply very closely to those in a law case, a surgical operation, or a business problem. In each of these cases the basic reasoning is often substantially this: "If A and B are true, then C is true, and from these facts it follows that D is also true." In law, "If A and

B have told the truth, then C's story must be true, and these lead to the conclusion that D's testimony is reliable." In surgery, "If the patient's symptoms are A and B, then we must expect the reaction C, and if this is the case we may be quite certain that D is to be expected."

10. Every lesson plan should look to the future as well as to the present. It should definitely plan for welding a link, but it should also see the chain. For example, a case in factoring may be developed and made the subject of drill; but the teacher should see it in relation to its possible use in fractions and to its later possible use in quadratic equations and, if far-sighted enough, to its use in the higher theory of equations.

11. A lesson plan should provide for a brief summary. It may be indicated by the words "Let us now see what we have learned today that is new to us and that we can use tomorrow." It means a great deal to all of us to "take stock" at the end of a day and see what we have done for the world. Perhaps the poet was thinking of a lesson or of a lesson plan when he wrote,

> Count that day lost whose low descending sun
> Views from thy hand no worthy action done.

Children like to summarize their work just as much as teachers like to "take stock."

12. It is advisable for beginners to write their lesson plans, even if these include only a few lines of brief suggestions. Many teachers continue to do this year after year, taking great pains to make note of the vital points of the next day's lesson. The plan is a good one, but there are some minds that can make and retain a set of mental notes that serve the same purpose. Whether to adopt the written or the mental plan depends upon the teacher's habit of recall. For most persons the written plan is the safer.

4. The Art of Questioning

The Purpose of Questioning. In a work on the teaching of mathematics it is impossible to give any elaborate treatment of such general educational topics as the art of questioning. It is proper, however, to treat briefly of a few points of major

importance, relating each to the work of the teacher of the mathematical sciences.

The purposes of questioning may be summarized briefly as follows:

1. To find what the pupil knows about the topic being studied. In this way the teacher is able to prepare him properly for the subsequent work.

For example, before teaching any law of exponents the questioning would need to show that the significance of the positive integral exponent is clearly understood.

2. To find such wrong ideas or difficulties as may hinder the next step in the pupil's progress.

For example, in the case just cited the preliminary questioning should reveal any wrong idea or difficulty arising from confusing coefficient with exponent.

3. To find, at the close of a development lesson, whether or not the teaching has been effective. In this way a teacher examines himself through his examination of the effect of the work upon his pupils. This is of great value to any teacher in improving his instruction.

Questions in a Lesson Plan. A good lesson plan should contain about eight or ten of the most searching and stimulating questions that the teacher can ask in connection with the subject under consideration. Naturally it is impossible to anticipate all that should be asked; indeed, this is fortunate, since otherwise the work would become very mechanical. It is better to have a small number of carefully considered questions than a large number of rather thoughtless and insignificant ones. Experiment has shown that a large number of questions is usually evidence of poor teaching, although it is evident that a small number is by no means an evidence of good teaching.

The best type of question is the one that appears to be a spontaneous request for desired information. This is the reason why pupils' questions are often better than those of a teacher. Modern education approves of such questions and the teacher does well to encourage the class to ask them. If someone in the class can answer a given question, the interest is thereby in-

creased. Usually, with a little help from the teacher, the answer will be forthcoming; if not, the teacher should find it a privilege to give the answer and the class will find it a pleasure to receive it. Questions asked simply "to kill time" are usually best ignored.

General Plans of Questioning. Various writers have laid down general methods of questioning, but for our present purposes only a few need be mentioned, as follows:

1. *The method of Socrates.* Ancient Greece developed some remarkable teachers. One of the greatest of these was Socrates, who lived about 425 B.C. His method was, through questioning, to convince a pupil of his relative ignorance and of the fact that his judgments were not well considered. He felt that thus he brought the pupil into a proper frame of mind to receive instruction. The difficulty of the method lies in the fact that it tends to discourage pupils and to lead a teacher to seek to exaggerate his own wisdom. It humbled the pupil but it did not serve to impart knowledge.

As a method for use today it has more dangers than virtues. It should not be a pleasant thing to show that others are always wrong.

2. *The question-and-answer method.* This has proved to be the best general type of instruction. If the answer is incorrect, it is usually easy, in a kindly spirit and without faultfinding or "nagging," to ask another question that will set the pupil right. If the answer is correct, it is usually easy not only to commend the pupil but to build upon that answer in the next question asked, thus forging a chain of instruction. If, at the same time, the teacher encourages fair questions from the pupil who really wishes further information, the lesson is likely to have more interest and the result is quite certain to be salutary.

Negative Suggestions on Questioning. The following negative suggestions may be of value:

1. *Do not depend upon the textbook.* Word the questions as you would naturally ask them, independently of the language of the text. Accept any answer that shows the pupil's under-

standing of the meaning without regard to the language of any author. If the language is not succinct and clear, suggest any change that is of real consequence, but do not be concerned to make trivial changes.

2. *Do not adhere to a regular order.* That is, do not follow the order of your roll or of the class seating; such a plan encourages inattention. Similarly, do not follow the exact sequence of the textbook.

3. *Do not call the name of the student before the question is asked.* To do so is to invite the inattention of the others. State the question clearly, and do not hesitate to call upon an inattentive student to answer it.

4. *Do not always call upon single individuals.* A question addressed to the whole class stimulates the attention and the thought of all. Call for the answer from among those who first raise a hand.

5. *Do not feel that you must confine yourself to formal questions.* For example, it serves a useful purpose to say, "State your reasons for believing that a square is at the same time a rectangle and a parallelogram."

6. *Do not form the habit of asking elliptical questions.* This means that we should avoid such incomplete forms as, "A triangle is a figure with three ——?" There are few more pronounced evidences of poor teaching than the continued use of this type. In a written completion test such types are legitimate, but in spoken questions they are usually stupid and they are never pleasing to the ear.

7. *Do not ask alternative questions.* For example, "Is this a triangle or is it a quadrilateral?" is usually a poor question, particularly as we may look upon every triangle as a quadrilateral with one angle of 180° if we desire to do so.

8. *Do not unduly hurry a pupil.* Remember that in a class of twenty there are precisely twenty different minds. Some pupils answer almost instantly, and often very thoughtlessly; others have slower mental processes, but not necessarily inferior ones. You know the answer before you ask the question, but the pupil does not, and he is not going to know it any more quickly if he

is at once subjected to a nervous strain that is unwarranted. What constitutes a reasonable time allowance is, of course, for the sympathetic teacher to determine.

9. *Do not hesitate to shorten a difficult question.* If a class has difficulty with a question, the fault is generally with the question itself, and the teacher should simplify the language or possibly break the question into two parts.

10. *Do not repeat a pupil's answer.* The single exception to this rule is found in the occasional desire to emphasize the statement. The mannerism of certain teachers of repeating every answer is annoying to both pupils and supervisors.

11. *Do not ask for collective answers.* As a rule only part of the class will reply, and in any case the weaker ones will wait until they hear the answer begun by the rest.

Types of Good Questions. Aside from the suggestion already made that the best question is a natural one, the following will be found helpful:

1. *A good question stimulates reflection.* For this reason a question that can be answered by a single word is usually to be avoided; for example, "Is a square a parallelogram?" is not the best type of question; it requires no reflection, because the answer is either known or it is not. The contrary is true when a teacher asks, "What reason have you for believing that $x^2 - 1 = 0$ is a quadratic equation?"

2. *A good question should be clear.* This requires that it be stated succinctly, that it involve no ambiguity, and that it be pointed. For example, "Why may we say that an equilateral triangle is isosceles?" is a good question; but "Is a quadrilateral a rectangle?" is a poor one, since sometimes it is and sometimes it is not.

3. *A good question is adapted to the understanding of the pupil.* For example, "Is the nature of a right triangle factual information?" would be a poor question for any pupil because he could not be expected to understand it; indeed, it would be a poor one for any adult because it is pedantic, — that is, it uses a poor English expression merely for the purpose of impressing the hearer with the wisdom of the questioner.

4. *A good question should be definite.* For example, "How should you proceed to deposit money in a savings bank?" is a good question; but "What do you know about logarithms?" is a poor one. The pupil might properly answer, "Nothing," and then claim full credit for answering the question correctly.

Types of Good Answers. The following suggestions will be found useful in determining acceptable answers:

1. *An acceptable answer should be complete.* For example, the reply to the question of when a quadrilateral is a parallelogram is not properly "When its sides are parallel," for this reply is both inaccurate and incomplete.

2. *It should be stated in good English.* There are many mathematical abbreviations like "tan," "cos," "cot," and so on, which are convenient in written work, but which should be ruled out of oral answers as mathematical slang.

3. *It should be in the pupil's own language.* This means that a memorized statement, whether a definition or an explanation, has little value in comparison with a statement from the pupil showing that he understands what he is saying. A word of commendation from the teacher is an appropriate reward for a clear reply of this kind.

5. Classroom Supervision

Purpose of Supervision. The purpose of supervising the work of the teacher in general, and the teacher of mathematics in particular, is to accomplish the following results:

1. *To encourage the teacher.* Supervision of teachers, like the supervision of the class by the teacher, is not primarily a matter of adverse criticism. Only a perverted mind is always seeking the bad. Supervision is primarily for the purpose of encouraging the one supervised, be he pupil or teacher or supervisor or principal, to do his best.

2. *To suggest possibilities for bettering the work.* Faultfinding is a negative action; it seldom helps anyone. On the other hand, an encouraging word, followed by suggestions based upon the experience of the supervisor, is always helpful.

3. *To suggest the best educational literature.* This does not mean a few ordinary textbooks on mathematics, but such works as those written by original scholars — men who know mathematics, its history, its range of applications, and its high-grade educational literature. There is so much mediocre writing in education that a supervisor has a duty and a privilege to make known the few high-grade, scholarly treatises upon the subject. A few such works are mentioned on pages 299–300.

4. *To suggest community interests.* In all subjects, and not the least in mathematics, the supervisor should suggest the bonds between the school and the community. For example, if a class is studying bank customs, it is very desirable to make contact with some local bank and have the results made real to the class through the report of a committee of pupils, a short talk by someone from the bank, or a conversation period with the class.

Duties of a Supervisor. In addition to the duties suggested under the study of the purpose of supervision, the following items will be helpful:

1. *Suggestions as to routine.* The supervisor should make suggestions to beginning teachers for expediting roll call, for proper seating, for ventilation, and for general classroom management.

2. *The work to be accomplished.* He should help the teacher, when necessary, in laying out the work for the term. In general, it is better to follow the textbook to a certain designated page, than to mix up the material to satisfy certain unimportant details of local theory.

3. *To encourage lesson plans.* In general, the encouragement is all that is needed. After that the decision lies with the teacher. For the supervisor to impose his own theories on the teacher is poor practice. In general, too, the supervisor should seek to relieve the teacher of all unnecessary burdens rather than to impose such burdens upon anyone, — teacher or pupils. Teachers labor, at the best, under a severe nervous strain, and this should not be unnecessarily increased.

4. *To meet with the teachers regularly.* Here too the purpose should be one of constructive criticism, — plans for betterment

through personal initiative. The supervisor who attempts to impose an array of unimportant detail upon his teachers, or who stifles all the originality that they possess, will find his efforts unproductive of any good results.

5. *To record the relative standing of teachers.* This is the most dangerous duty imposed upon supervisors. The knowledge that an official who is very human, and who may not be particularly better as a teacher than those observed, is making a mathematical calculation of a teacher's ability, is enough to disconcert anyone who is conducting a class. Only the best kind of spirit on the part of the supervisor can make the ordeal endurable to any teacher of fine sensibilities. If, however, the task results in constructive aid and in a reliable report to the principal, much value may come from the effort. Since the question is general rather than one which relates to mathematics alone, the teacher is referred for more detailed information to the literature upon the subject.

QUESTIONS AND TOPICS FOR DISCUSSION

1. What do you consider to be the contribution of Herbart so far as the inductive-development lesson is concerned? What are the dangers encountered by its use?

2. Where and under what conditions do you think the inductive-development lesson should be used?

3. Write a plan that you may later use in teaching an inductive-development lesson on some topic in mathematics. State ten of the most important pivotal and thought-provoking questions you would ask in teaching that lesson.

4. What can be done to increase the list of natural questions in the classroom and to minimize the formal ones usually asked by the teacher?

5. Where should you make large use of the deductive type of lesson?

6. Write a plan for teaching a deductive-development lesson on some topic in mathematics.

7. Write a plan for teaching a drill lesson on the laws of exponents in multiplication.

8. Show how you would proceed in planning a review lesson on the geometry of position.

9. Write a plan for an examination lesson that you may use at the end of your teaching of a unit of numerical trigonometry.

10. Write a list of attainments, including training and experience, that should characterize the head of a department or a supervisor of mathematics.

11. Discuss clearly and fully the place, nature, and purpose of the lesson assignment.

12. Give a critical evaluation of the nature and purpose of supervised study.

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CHAPTER X

MODEL LESSONS

1. General Purpose of the Chapter

Purpose Stated. Teachers who are just beginning their work often say, and with much reason, that books on teaching talk a great deal about how things should or should not be done, but that they do very little in the way of showing how the authors would actually teach a class themselves. It is the purpose of this chapter to meet this criticism. A few model lessons will therefore be given in some detail, in the hope that they may prove suggestive to those who stand in need of such help.

Difficulties in the Attempt. A book with an extended series of model lessons would however be like an old-style letter writer; it would tend to contain a mass of formalism that would take away all a teacher's initiative. For a good teacher it would be unnecessary, while for a poor one, who might attempt to follow it too closely, it would be even harmful.

No one would ever think of giving a model of an hour's conversation, and this is what a class period in mathematics should be. Directions for a model way of walking with a friend or of conversing at a dinner would be equally ridiculous.

Those things which combine to make a good teacher, — his personality, his enthusiasm, his unconscious control of a class, his ability to avoid the trivial, his broad learning, his succinctness of expression, his power to anticipate difficulties, his pleasing voice and manner, his sympathy with youth, and his ability to keep the purpose of the lesson in view while directing the discussion, — such elements show themselves but slightly in a model lesson, although they are the characteristics of a model teacher. In spite of all this, however, to a teacher who is just beginning his work there is a certain value in considering even

a fictitious presentation of the critical points and some of the discussion in a well-conducted lesson.

Purpose of the Proposed Lessons. It is now proposed to give the outline of a lesson in algebra in Grade IX or earlier. For teacher and pupil alike the purpose is to show how to solve an equation of the type $3w = 27$, thus preparing for the derivation of one formula from another, as in the case of deriving a formula for w from the formula for the area of a rectangle, $A = lw$. Since only one lesson relating to algebra is given, this is made longer than it would be in actual teaching.

2. A Lesson in Algebra (Grade IX)

Introductory Statement by the Teacher. We have seen that one great value of algebra lies in its use as shorthand. It shows us how to write a rule as a formula. We have learned that the formula for the area of a rectangle is $A = lw$. Another great value of algebra is that we can easily obtain one formula from another without having to remember it. The more we study it the more we shall come to see that algebra makes mathematics easy and interesting.

Today we shall see how to obtain one kind of formula from another.

First, let us take the formula just mentioned, $A = lw$. What does this formula tell us?

PUPIL. It tells us that the area of a rectangle is the product of the length and width.

TEACHER. Does it mean that you multiply one line by another?

PUPIL. No, it means that the number of square units of area is the product of the numbers representing the length and width. It is only a short way of saying all this.

TEACHER. If, for the formula, I give you the value 12 for A and 3 for l, what does the formula become?

PUPIL. The formula becomes either

$$12 = 3w$$

or

$$3w = 12.$$

TEACHER. Yes, that is right; you may express the formula either way, but what is the use of doing so?

PUPIL. We have found that sometimes it is easier to use it in one of these forms than in the other. We have a right to give both forms; for if $2 \times 3 = 6$, then $6 = 2 \times 3$.

TEACHER. Do any of you remember what name we give to the two sides of a formula? [Calls on one of those whose hands are up, and does the same in subsequent cases of this kind.]

PUPIL. If we write the formula as $3w = 12$, then $3w$ is called the "left side" and 12 is called the "right side." We may also speak of "member" instead of "side" if we wish.

TEACHER. Let us now see what is the value of w. Who can tell us what it is?

PUPIL. If $3w = 12$, then $w = 4$.

TEACHER. That is correct, but how did you find it?

PUPIL. If $3w$ is 12, then one w must be $\frac{1}{3}$ of 12, or 4.

TEACHER. Yes, but if the formula had been $1\frac{1}{4}w = 12$, it would not have been so easy to tell how you found the value of w. What should you have done then?

PUPIL. I should have divided 12 by $1\frac{1}{4}$.

TEACHER. Yes, that is right, and so you have found the general way of solving in such a case. If $3w = 12$, we find w when we divide $3w$ by 3; but we must then divide 12 by 3 so as to keep the equality true.

PUPIL. Do you always have to divide both sides by 3?

TEACHER. What do you think about it? Suppose that we had $5w = 30$, should you divide both sides by 3?

PUPIL. No, I should then divide both sides by 5.

TEACHER. Why do you take 5?

PUPIL. I take 5 because there are five w's.

TEACHER. That is right; but why do you divide both sides by 5?

PUPIL. I divide both sides by 5 so as to keep the formula equally balanced; it keeps the equality if I divide both sides by the same number.

TEACHER. That is a good statement. Now I want you to know that a formula like $3w = 12$ is also called an *equation*;

that in this case $3w$ is called the "left side," the "left member," or the "first member." What do you think the 12 is then called?

PUPIL. The 12 must then be the "right side," "right member," or "second member"; but do we have to remember all this?

TEACHER. No, it is just as well at present to speak of the "sides" of an equation or formula, but if you wish to speak of "members," as we often do in algebra, we shall all know what you mean.

How do you know that 4 is the value of w in the equation $3w = 12$?

PUPIL. I know that $w = 4$ because $3 \times 4 = 12$.

TEACHER. Yes, that proves that you are right.

When you find that $w = 4$ we say that you have *solved* the equation $3w = 12$, and we call 4 the *root* of the equation.

When you prove this by seeing that $3 \times 4 = 12$, we say that you *check* the root.

The finding of the root is called the *solution of the equation*, and so we also say that we check the solution.

PUPIL. Can we always find w by dividing the area by the length?

TEACHER. Yes, if we understand that this means dividing the value of the area by the value of the length. We cannot very well think of dividing a piece of paper by an edge, but we can easily think of dividing the number (say 12) that represents the area of the paper by the number (say 4) that represents the length.

Now suppose that I tell you that I am thinking of a number which, when multiplied by 4, becomes 32; how should you state this as an equation?

PUPIL. If we use the initial n for number, we have the equation $4n = 32$.

TEACHER. That is correct. We may now say that we have translated the statement into *algebraic shorthand*. The equation which you give is a short way of saying that 4 times some number is 32, or of asking the question "For what value of n will $4n$ be equal to 32?" [Calls upon one of the pupils to write the equation on the board.] Who can now answer the question; that is, who can solve the equation?

PUPIL. I can solve it; n is equal to 8.

TEACHER. How did you find the answer?

PUPIL. I divided both sides by 4.

TEACHER. How do you know that this is the correct answer, or the root of the equation?

PUPIL. I know that it is correct because $4 \times 8 = 32$.

TEACHER. Yes, you have checked the answer. Now all of you may write on paper the equation $8n = 28$ and then solve it.

PUPIL. I think that $n = 3\frac{1}{2}$.

TEACHER. Why do you think so?

PUPIL. Because I divided both sides by 8, and this gave me $n = 3\frac{1}{2}$.

TEACHER. Did you check the result?

PUPIL. Yes, I checked it by saying $8 \times 3\frac{1}{2} = 28$.

TEACHER. Then should you say, "I *think* that $n = 3\frac{1}{2}$"?

PUPIL. No, I should have said, "I *know* that $n = 3\frac{1}{2}$," because I have checked the work.

TEACHER. That is correct. You probably see that, in checking, it is easier to think of the answer as $\frac{7}{2}$, because it is easier to find $8 \times \frac{7}{2}$ than $8 \times 3\frac{1}{2}$. When we write the answer, however, it is better to write $3\frac{1}{2}$ or 3.5, because these are more familiar forms than $\frac{7}{2}$.

Do you all now see how to solve equations like $5n = 45$, $7n = 91$, and $12n = 30$? If so, copy the following equations:

1. $2w = 16$.	5. $7w = 14$.	9. $3b = 10$.
2. $3l = 15$.	6. $9n = 18$.	10. $4b = 10$.
3. $8n = 32$.	7. $9n = 21$.	11. $6c = 20$.
4. $5b = 25$.	8. $8n = 36$.	12. $10p = 95$.

Now when I say "Begin," write after each equation the value of the letter. You will have just 2 minutes to do this, checking each result mentally.

PUPIL. [The results are read by the pupils and the errors are discussed.]

TEACHER. Suppose that you have the equation $36 = 3n$, how will you solve it?

PUPIL. I should solve just as I should if it were written $3n = 36$, and I should find that $12 = n$, or $n = 12$.

TEACHER. That is right; the result may be stated in either way, but we usually change the equation around so as to read $3n = 36$, and we then say that $n = 12$. Now copy these equations and solve them as rapidly as you can, checking each result:

1. $15 = 3n$.	5. $40 = 10a$.	9. $25 = 2a$.
2. $18 = 3p$.	6. $50 = 5b$.	10. $35 = 10c$.
3. $21 = 7w$.	7. $55 = 11b$.	11. $4 = 3m$.
4. $35 = 5t$.	8. $60 = 10l$.	12. $40 = 30m$.

TEACHER. [After discussing the equations and solutions.] We may also have occasion to solve equations containing decimals. For example, the length of our national flag is defined by government regulations to be 1.9 times the height (or width). Will someone tell us how to state this as a formula?

PUPIL. The formula is

$$l = 1.9h,$$

because the length l must be 1.9 times the height h.

TEACHER. Now suppose that we wish to make a flag 8 inches long, how high (or wide) will it be?

PUPIL. We must write 8 in place of l and solve the equation.

TEACHER. That is correct. Now you may all write the equation. Before we solve, let each of us estimate the approximate answer, which you can do by noticing that 1.9 is nearly 2. One of the best checks that we have is this of estimating the answer before we begin to solve. Now write the estimate. [A pupil writes 4 on the board.] We shall now solve the equation at the board.

If $$1.9h = 8,$$

then $$h = \frac{8}{1.9}$$

$$= \frac{80}{19}$$

$$= 4.21.$$

Practically, if we were making the flag, to how many decimal places should we carry the result?

PUPIL. I should think to tenths. Our rulers do not give hundredths of an inch, but they give eighths, and so we can estimate easily to tenths.

TEACHER. I read the other day that an art club had suggested to the government that the flag would be more pleasing to the eye if we used 1.6 instead of 1.9. What would our equation then become?

PUPIL. It would become $8 = 1.6\,h$, or $1.6\,h = 8$. This would be easier to solve, for we should then have $h = 5$.

TEACHER. Let us now solve a few equations with both common and decimal fractions.

First write the following equations:

1. $2.1\,n = 4.2$.	5. $\frac{3}{5}\,x = 9$.	9. $3\,a = 10$.
2. $3.6 = 1.2\,n$.	6. $8 = \dfrac{4\,a}{5}$.	10. $20 = 6\,n$.
3. $6\,x = 3.6$.	7. $\frac{5}{8}\,t = 15$.	11. $20 = \dfrac{2\,n}{3}$.
4. $60 = 0.6\,x$.	8. $49 = \dfrac{7\,n}{8}$.	12. $\dfrac{3\,a}{8} = 4$.

Now write after each equation your estimate of the approximate answer, doing this in 2 minutes.

Now solve each, taking 5 minutes for the work and the checking.

The answers should then be read and compared with the estimates. The difficulties that have developed in dividing by decimal and common fractions should be discussed at the board. The errors should furnish a basis for the work of the next day.

Now let us come back to the formula. We see that we may write

$$A = lw$$

in another way; that is, as

$$lw = A.$$

Dividing both sides by l, we have

$$w = \frac{A}{l}.$$

That is, we have found a formula for w.

Now in the following cases find the formula for the letters specified:

1. $A = lw$; for l.
2. $A = bh$; for b.
3. $A = bh$; for h.
4. $C = \pi d$; for d.

3. A Lesson on Other Spaces (Grade VII)

Reasons for Teaching the Lesson. It is doubtful whether anyone ever really liked a lesson because the teacher assured him that he would have use for it at some later time. To expect that a pupil would be much impressed by such a statement would be like expecting him to become enthusiastic about reading a dictionary. Merely to place a textbook, even of the best modern type, in the hands of children and to tell them to read it and admire it is about the poorest way we could invent for teaching any subject whatever. Geometry, for example, should not be taught by merely requiring the reading of a textbook, and much less the memorizing of it. Such a book should be looked upon as a "spiritual kaleidoscope." If we merely look in a kaleidoscope, the picture has only a moderate amount of interest; it is only as we give it motion that its beauty appears.

One of the methods of improving upon mere textbook teaching is to call attention to the wide use of mathematics, a fact which is apparent to everyone. If, however, we seek to reveal all these uses to a body of pupils, we fail through the mere superabundance of material and through the lack of technical knowledge necessary to make use of what is easily available.

If we try to make the material seem less technical by means of class play, as in playing store or bank or machine shop, we run into one of the danger zones of teaching, — the wasting of valuable time and the loss of that rigorous thinking which forms a vital part of the subject taught. A reasonable amount of this work is justifiable, or even highly commendable, particularly in the elementary school, but in the junior high school it is especially essential to avoid any extreme in this field. "All work and no play" is a dangerous extreme, but "All play and no work"

is more so. The ideal mean which we should all seek to approach may be indicated by the words "All work should be play," — that is, it should be made just as interesting as possible.

To attain this mean we need to make the kaleidoscope move, to show to the pupils the beauty, the grandeur, and the general significance of each mathematical topic, — to show the soul as well as the body, and the body as well as the mere clothing. To do this we must give to the children a freedom of play of the imagination that was generally lacking in the older type of teaching, — a subject referred to elsewhere in this work. What Coleridge said about mathematics, that "whilst Reason is luxuriating in its proper paradise, Imagination is wearily traveling on a dreary desert," represents the older type; the modern type allows imagination an equal place with reason in the oasis. It is only in this way that we can remove the charge that mathematics is mere drudgery to many children.

There is a suggestive story told of a small boy in a drawing class. Asked to draw a picture of a man, he represented him with a very large head and with a single leg. When the teacher remonstrated with him and pointed out the defects, the reply was confidently made, "Well, that's the kind of man he is." In a large measure the child was right; he had given free rein to his imagination and to his own originality of expression.

It is evident, however, that the textbook cannot go very far in suggesting the proper play of imagination. To do so would be to tell too much, and thereby to eliminate the very thing that the teacher may properly encourage. We therefore give this lesson to show how the teacher may proceed in one of the many fields which intuitive geometry opens to the pupils. The lesson is one which is given every year in one of our schools of observation and may be entitled "Other Spaces than Ours."

Introductory Statement by the Teacher. Today we are going on another little journey, this time to other spaces than ours. We are going to let our imaginations discover what it would be like to live in worlds of such spaces.

Let us first imagine that we all tried to live in "Pointland,"

— a land consisting of only a single point. Since none of us could be smaller than a point, and we think of Pointland as having only one point, we probably will think of only one of us in such a land. Each of us would then fill the whole of his universe. We think that we have but little room in a crowded city, although we have three dimensions (length, breadth, and height) in which to move; but in Pointland we could not move at all, although we should not realize it. Each would think of himself as the whole universe. The more narrow-minded a man gets the bigger he thinks he is. Mr. Pointlander, therefore, could not move, could not play any games, could not read or eat or drink, could not do anything that we do; for doing involves motion and he could not move at all. We can roughly picture him like the head of a pin driven into a board and the hole plugged up.

Now let us suppose that Mr. Pointlander becomes ambitious, as we often do ourselves; that he discovers that there is another universe in which he would like to move, and that he is able to break away from his no-dimensional space into one that lies just a stage beyond Pointland. What do you think that land would be called?

PUPIL. That would have to be the land of a line.

TEACHER. Yes; in other words, Lineland, — the path of a moving point. What could Mr. Pointlander now do that he could not do before?

PUPIL. He could go in two opposite directions only, and would be confined to a straight line.

TEACHER. You are right. He would now seem very free as compared with his former self, because he could go forward and backward at will. Nevertheless, he would be much limited in his movements. If there were three points, 1, 2, and 3, in this order, Point 1 could never be next to Point 3, because Point 2 would always be between them. A person in Lineland would, if enormously magnified, be very much like a fly in a lemonade straw or a man crawling in a sewer pipe that was just large enough for him to get through, each being free to move only forward and backward.

Now that our Pointlander has been given some length and has thus been transformed into a Linelander, let us imagine that he becomes endowed with still greater intellect and with higher reasoning powers. What will his ambition lead him to discover next?

PUPIL. I think he would discover us.

TEACHER. Perhaps so, but not if he were logical in his thinking. There is another land between our land and Lineland. Can you tell what it is?

PUPIL. Surfaceland.

TEACHER. That is a very good answer. Some writers have called it Flatland, and that is what we shall call it. Do you see that when a straight line moves in any direction except along itself it traces out, or generates, a surface like the plane of the blackboard?

PUPIL. That is easy to see. It is like this.

The pupil takes a yardstick and illustrates by moving the stick sideways through the air.

TEACHER. We can see that in Flatland our former Pointlanders and Linelanders would feel that they were very fortunate. They would be as happy as Alice in Wonderland, free to move sideways as well as straight ahead, and therefore able to pass one another. And yet, in spite of all this, they would seem to us very much handicapped. For example, they could see nothing outside the plane in which they lived, being quite like our shadows as they appear upon the floor or a wall. Indeed, their universe might properly be designated as Shadowland. Their houses would look like our letter T; for any doors would shut out all air and light, and they could be opened only by pushing them over on the line that makes the Flatlander's ground. The people might be described as resembling somewhat a bee between two parallel panes of glass, or like rafts on the surface of a lake.

They would be as safe from observation by each other behind a line as we are behind a wall. We could see them, but they could not see us. If their Flatland were horizontal to us, and we should call to them, "Look up," the idea would mean nothing even if

they understood the words, for they would have no idea of the significance of "up" or "down."

If there were a savings bank in Flatland, the money would be safe if a circle that could not be broken were put around it. Do you see why the circle might have to be a very large one?

PUPIL. The dollars could not be piled up as in our banks. They would have to be spread around.

TEACHER. Exactly so, if the dollar bills were like ours; but if we, in three dimensions, use two-dimensional bills, what would the two-dimensional people probably use?

PUPIL. They might use one-dimensional bills, and if they laid these side by side it would be much like our piling up bills.

TEACHER. Yes, I see that you have imagination. Let me ask you another question. How should you tie a knot in Flatland?

PUPIL. I cannot imagine Flatlanders being able to do it.

TEACHER. You are right. The Flatlander has much more freedom than the Linelander, but nothing like our freedom, because, for one thing, he cannot tie a knot.

If the lesson were given to older pupils, the teacher might add that in Flatland the teacher could not use the method of superposition, for it would be impossible to pick up a triangle and place it on another. All that could be done would be to make two points or two lines coincide.

TEACHER. If the Flatlander's universe were the still surface of a body of water, and if we should put the vertex of a cone on this surface, he would see a point. If we should push the cone downward into the water, the Fatlander would see a circle which would grow larger and larger. To him it would seem as it would seem to us if a point began to grow into a sphere, this becoming larger and larger. If the cone were finally allowed to sink, Mr. Flatlander would be surprised to see the circle suddenly disappear. It would be quite as if we saw a sphere in the air before us which, after increasing in size, suddenly disappeared altogether.

A Flatlander could enter or leave Lineland at any point, consciously or unconsciously, just as you or I would leave Flatland; but otherwise he is very much handicapped, at least from our point of view. But now, with his improved reasoning power, let

us suppose that he sets his imagination free to imagine higher universes and that he longs to be transported to a world of one more dimension. What world or universe would that be, and who lives there?

PUPIL. That would be our universe and he would then be one of us.

TEACHER. Yes, but think how he would appear to his former friends, the Flatlanders. He could at once do wonderful things in the eyes of these old acquaintances. He could vanish from Flatland and return at will. He could not be held in the strongest prison in their land. He could see into his former neighbor's house and even into his brain. If he preserved his flatness but turned over in our land, he would appear to his former friends, when he returned to Flatland, as a sort of reflection of his old self, just as one of our gloves is a kind of reflection of the other. If he had been right-handed before, he would now be left-handed; if he had had a scar on the right cheek before, it would be on the left cheek now; and if his heart had been on the left side before, it would now be on the right side.

PUPIL. If he lay face upward before, I should think he would lie face downward now.

TEACHER. Yes, but that does not matter to a Flatlander. His face is on the bounding line, it is not a surface. The face line would simply be on the other side of him. If all faces normally turned one way in Flatland, everyone would notice him, for his face would turn the wrong way.

Do you now see what it has meant to our imaginary being to be transferred from a land in which he was like the head of a pin driven into a board to a land where he would be like a fly in a lemonade straw, and then to a universe where he would be like a thin spot of oil on the smooth surface of a lake; and finally to our universe, where he would be like an airplane that is free to move in three dimensions?

PUPIL. I should think he would then wish to go still farther.

TEACHER. Let us suppose that he does, assuming that his ambition carries him into a universe of four dimensions. If this were the case, let us imagine what he might do. Just as we per-

form some very strange acts in the eyes of the two-dimensional shadows of Flatland, so he will perform some strange ones in our sight. For example, just as we can step into the Flatlander's house or prison without passing through the line-walls, so this creature in four dimensions could step into our houses or prisons or safe-deposit vaults without passing through the plane-walls. He could tie a knot in a string and then pull it out even if the ends were fastened.

If such a person played on the football team at his school, he could turn a football wrong side out without cutting it. Turning a right-hand glove so as to make it fit the left hand would be simply like turning it over in the fourth dimension, just as the flat drawing of such a glove can be turned over in three dimensions.

A fourth-dimension surgeon would be able to reach the innermost part of the body without making an incision, and having one's appendix taken out would be much like cutting a corn. Moreover, the surgeon would be able to see how much money his patient had in his pocket, or even in his safe-deposit box, and could take it without the formality of sending a bill; and the butcher and grocer and dry-goods merchant from that universe could do the same, and so could we, in that universe, do it in return. The result would be that, if we had fourth-dimensional beings about, we should have to alter greatly our plans of living.

All this is imagination, but it is interesting and it leads us to appreciate a term that is much used in astronomy at the present time, — "fourth dimension." Whether such a dimension exists or not, the speculation sets us thinking more seriously of our position in the great universe about us. The possibility is perhaps no more remote than was telephoning across the Atlantic by wireless a few years ago, or sending pictures by radio, or crossing the ocean by airplane, or seeing at a long distance by the aid of electric waves.

It is profitable to allow the pupils to draw pictures of beings in Lineland and of those in Flatland, with the houses of the latter. This will help them to get an idea of the various kinds of plane figures which are studied in geometry.

4. A Lesson on the Geometry of Size (Grade VII)

Introductory Statement by the Teacher. If we are asked to find the size of a certain plane figure, we say that we are measuring its *area*. As in measuring length, the process is one of comparison. We compare the area of the given polygon with some defined and accepted (standardized) unit of area and thus determine the size of the polygon. In order to do this we have to find the number of times the area of the polygon contains some standard unit of area. Does anyone here know what the standard unit of area is?

Pupil. I think it is the square foot.

Teacher. That is a fairly good answer, but, as you all know, the unit of area is not always a square foot. Sometimes it is a square inch. We can say, however, that the unit of area in any case is always the area of a square, each of whose sides is a standard unit of length. Thus, we may measure area and express the result in square inches, square feet, square yards, square centimeters, and square meters according as the standard unit of length is an inch, a foot, a yard, a centimeter, or a meter.

Does anyone in the class know a practical way to estimate the area of a polygon?

Pupil. We did it in the lower grades by using tracing paper. We transferred rectangles to squared paper and then found their area by counting the number of squares inclosed by the figure.

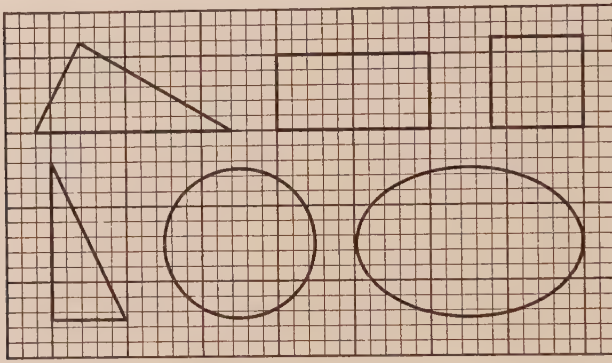
Teacher. That is one way to do it. However, we must remember that if the lines of a figure happen to cut the squares it becomes necessary to approximate the result. In such approximations we have to be careful not to go beyond reasonable limits of accuracy.

The six figures you see in your text (see the following page) were transferred by means of tracing paper to ordinary squared paper ruled to the metric scale. Study each figure and be ready to give the approximate area of each in either square centimeters or square millimeters.

The teacher then asks various pupils to give their results; these results are checked and the teacher continues.

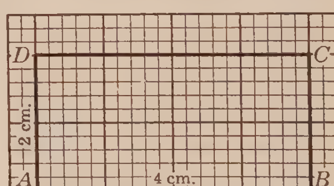
If the paper were ruled much finer should you be able to make a more accurate estimate of the area?

PUPIL. Yes, unless the squares were too small to be counted.



TEACHER. Look next at this rectangle (see below). The sides of the rectangle are integral (whole) multiples of 1 cm. Using 1 sq. cm. as a unit, there are two rows of units, and four units in a row. What is the area?

PUPIL. There are two times four, or eight, squares in the rectangle.



TEACHER. Yes, in other words we can say that

$$\begin{aligned} A &= 2 \times 4 \\ &= 8; \end{aligned}$$

or, stating this as a rule, *the number of square units in the area of any rectangle is the number of units in the length multiplied by the number of units in the width.*

This rule is usually shortened as follows:

The area of a rectangle is the length times the width.

We may further abbreviate the rule by writing the following shorthand:

$$Area = length \times width.$$

We may abbreviate even more by using initial letters and writing

$$A = l \times w.$$

We may go even further by writing a dot in place of $\times$ as is often done in algebra. We then have

$$A = l \cdot w.$$

If we now agree that when there is no sign between two letters in such an expression, it is understood that their values are to be multiplied together, we have

$$A = lw.$$

Such an expression is called a *formula* and it illustrates the main use of algebra, a subject in which we are principally concerned either with rules or with the formulas which are shorthand expressions of these rules.

Let us now see if we can find the area of certain rectangles when the length and width are given. For example, what is the area of a rectangle whose length is 8 in. and whose width is 3 in.?

PUPIL. The area is 24 sq. in.

TEACHER. How did you get it?

PUPIL. I multiplied 8 by 3.

The teacher then gives a few more simple cases like this to see if the pupils understand how to use the rule and the formula. He then asks the pupils to use the formula $A = lw$ to find the value of A when

1. $l = 8$ and $w = 7$.	4. $l = 6$ and $w = 2$.	7. $l = 15$ and $w = 10$.
2. $l = 9$ and $w = 4$.	5. $l = 2\frac{1}{2}$ and $w = 3$.	8. $l = 20$ and $w = 10.5$.
3. $l = 20$ and $w = 5$.	6. $l = 4.5$ and $w = 6.9$.	9. $l = 40$ and $w = 35$.

After the pupils have solved these problems in their notebooks and the results have been checked, the teacher explains to the class that when we know the values of l and w, as in the above examples, and use them to find the values of A, we are said to *evaluate the formula*.

TEACHER. To evaluate a formula we proceed as follows:

1. *Substitute the given numbers for their letters.*
2. *Reduce the result to the simplest form.*

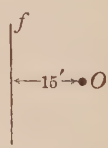
For example, if $l = 4.8$ and $w = 7.4$, we see by substituting these values for l and w that $A = lw = 4.8 \times 7.4 = 35.52$.

Tomorrow we shall continue our work on the evaluation of formulas.

5. A Lesson on the Geometry of Position (Grade VII)

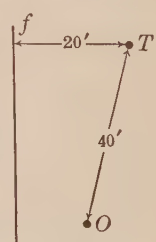
Introductory Statement by the Teacher. I read an interesting story the other day about an old pirate captain who lived in a village along the New England coast in the early colonial days. According to the stories told about him this old pirate had accumulated great wealth, — principally gold and other valuables, — but no one knew where the treasure was hidden.

The captain had no immediate relatives, so from time to time he lived around among various families in the village. It happened that there was one man in the town who had been unusually kind to him and with whom he was living when one night he was attacked and fatally wounded by a rival pirate crew. As he lay on his deathbed at the home of his friend, the old captain, seeing that his end was near, told the man how much he had appreciated the many kind deeds he had done for him, and that he wanted to do something in return; and he said: "I have no immediate family, I cannot take my treasure along, and I want you to have it. If you will go south along the fence separating your pasture from the cornfield, and continue until you reach the tall oak tree, you will find, 20 ft. from the fence and 40 ft. from the tree, the place where my treasure is buried. I give it all to you in return for your many kindnesses."



After the old captain had died, the man went to the old oak tree O and found that it was 15 ft. to the east of the fence f, as shown in the figure above. How do you think he proceeded to locate the treasure?

Pupil. If I were the man, I should take a rope about 45 ft. long, tie one end around the base of the tree so as to leave just 40 ft. that I could stretch taut. I should then place the other end at a point 20 ft. from the fence.

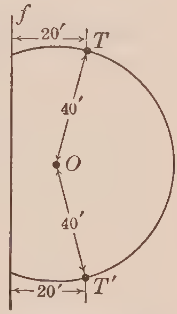


The pupil then draws this figure to show where he thinks the treasure T is buried.

Teacher. Do you think the man would find the treasure at the point you have thus located?

PUPIL. Yes, I think he would, because the point is 20 ft. from the fence and 40 ft. from the tree, as the captain said.

TEACHER. How many think that the man would have found the treasure where Mary says he would?



Some of the pupils think Mary is right, some are uncertain, and others think she is wrong.

PUPIL. Did he really find it there?

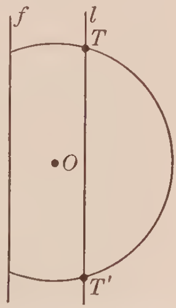
TEACHER. No, the story says that he did not find the gold there. He did just as Mary suggested, but the gold was not there.

PUPIL. Did the pirate lie about it?

TEACHER. No, the pirate's directions were all right, but the man was disappointed. He went home and told his son George what had happened. George had studied geometry in school; and so he told his father that according to the directions given by the pirate captain there were other possible locations for the buried treasure. Now where do you suppose the man looked next?

By this time the interest and excitement of the pupils is at high pitch, and the statement about George's having studied geometry leads them to suspect that geometry has something to do with the solution of the problem.

PUPIL. I think I know another possible location. It is just like Mary's solution except that it is farther south along the fence, and still 20 ft. from it.



The pupil then proceeds to add to Mary's figure the other possible location as shown in the figure above.

By this time other hands begin to go up, as the pupils see other possibilities.

TEACHER. Can anyone tell me on what two lines the treasure must lie?

PUPIL. It must lie on a circle with the center O and a radius of 40 ft. It must also lie on a line parallel to the fence at a distance of 20 ft.

The pupil then draws the second figure, above, on the blackboard.

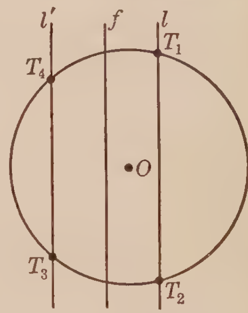
The lines meet in two points, and so there are only two possible places where the treasure can lie.

TEACHER. How should you go to work to locate these points if you were hunting for the treasure?

PUPIL. I should fasten a pointed stick to the end of the rope and scratch a circle on the ground. Then I should measure 20 ft. from the fence and scratch a line parallel to the fence. At the two points where this line crossed the circle I should dig and find the treasure in one place or the other.

TEACHER. Well, the man did just this, and yet the treasure was in neither place.

PUPIL. I know! It was on the other side of the fence! I can show you on the blackboard.



He sketches the figure here shown (not to scale), marking the possible points where the treasure could be found, — T_1, T_2, T_3, and T_4.

TEACHER. Yes, and you have done very well to think this out. There are two parallel straight lines that we know, and one possible circle. The points of intersection of the two parallels and the circle are the four possible places where the treasure may be found.

6. A LESSON ON THE ARITHMETIC OF THRIFT (GRADE VIII)

Introductory Statement by the Teacher. Today we are going to study some of the important qualities of a thrifty person and see what is to be gained by developing the habit of saving a little of our earnings each week, month, or year.

Some people have the idea that a person who saves his money is miserly. This may be true in some cases, but it is not necessarily or usually the case. Whether or not a man is a miser depends upon the way in which he saves and also upon the amount. If people had not earned and saved money, we should not have comfortable homes in which to live, mills and factories in which clothing and other necessities are made, railroads to bring food to the cities from the farm, or school build-

ings in which to obtain an education. All these things represent money which was saved by someone.

Now what is the result of not developing habits of thrift, of not saving a little of what we earn, and of not taking proper care of it?

PUPIL. We have nothing left for a "rainy day."

TEACHER. This means that we shall not have money when we need it most. This is the reason why you will be interested in this chart which I have drawn and which is based upon insurance statistics. We may call it a map of the march through life.

The bars represent the financial conditions of men at the ages of 35, 45, 55, and 65, all of whom had started in business life at the age of 25. The shaded sections at the left show the number who died before reaching the ages specified. The vertically-shaded sections (W) show the per cent who were wealthy or well-to-do, saving something each year. The white sections (E) show the per cent who just lived upon their earnings. The shaded sections (P) show the per cent that were either no better off than when they began or were dependent upon others for support.

This chart tells an interesting but pathetic story. What does it teach us?

PUPIL. It teaches that the percentage of wealthy or well-to-do people decreases rapidly after 35.

TEACHER. Yes, that is one thing. Some die and some lose their money through bad investments. What else does the chart teach us?

PUPIL. People who live on their earnings increase in number from 35 to 45, decrease from 45 to 55, and become very much less in number from 55 to 65.

TEACHER. Yes, that is also very interesting; but what other lesson can we learn from the map of "the march through life"?

PUPIL. The percentage of those who had not succeeded up to 35, and of those who were so poor as to depend upon others, decreased from the age of 35 to that of 45.

TEACHER. That is true. They had begun to find that they could do more to support themselves and to get into the earning class. What else does the map tell us?

PUPIL. The percentage of poor or dependent men increases very rapidly up to 65, and at that time 55% of those who began business at the age of 25 are in that class.

TEACHER. Yes, but that is not the worst of the matter. Not counting those who died, what per cent of those living at 65 are now in the poor class?

PUPIL. Counting the squares, I see that 13 squares represent the living, and 11 squares represent the dependent group, so that $\frac{11}{13}$ are poor.

TEACHER. You may show on the board what per cent are in this group.

PUPIL. [After working at the board.] The result is a little over 84.6%.

TEACHER. This means, then, that of 100 young men who began business life at 25, about 65% are living at the age of 65, and of these about 85% are dependent upon others. What does this teach us?

PUPIL. It teaches us that we should try to save at least a small amount of what we earn and to see that this money is invested safely for our old age, so that we shall not be dependent as so many old people are.

TEACHER. What does the class think would be a good per cent for each person to save from his monthly salary as provision for a "rainy day"?

Some of the pupils say "five," some "ten," some "fifteen," and some as much as "twenty."

George has suggested that the main thing is to "save something," and that is really the gist of the matter. James J. Hill, one of the most successful pioneers of the West, is reported to have said, "If you want to know whether you are destined to be a success or not, you can easily find out. The test is simple and infallible. *Are you able to save money? If not, drop out. You will lose, you may think not, but you will lose as sure as fate, for the seed of success is not in you.*"

This states the case very strongly, but there is much to be said for the idea expressed. I should say that we ought all to save at least 10%. If we were to agree on such an amount, how much would a boy save in a year if he had a monthly salary of \$50 and saved 10% of it each month?

PUPIL. He would save \$60, because 10% of 50 is 5 and $12 \times \$5 = \$60$.

TEACHER. Very good. Do you all know how such per cents are computed? If so, let us pass on to certain other simple and yet important matters of thrift. Some of them are very often overlooked even by supposedly thoughtful people. We should not be willing to be found in such a group. We all see by this time the importance of saving money, but does our obligation end even there?

PUPIL. No, we should learn how to take care of it. That means that we should learn how best to invest it or to put it where it will be safe.

TEACHER. Janet is correct, but there are some who think that in order to keep their money in a safe place, they should bury it in the ground. This was often done in certain countries during the World War and is resorted to by many people even in times of peace. What should you do with the money you saved if you were on a salary?

PUPIL. I should put my savings in a bank.

TEACHER. That is a good idea, but there are many different kinds of banks. Have you any particular kind of bank in mind that you would recommend?

PUPIL. Yes, a savings bank. I should put it there because I know a bank where my father receives 4% interest on all of his deposits.

TEACHER. How many think Paul has a good plan in mind?

The pupils are not entirely agreed. One boy thinks he would like to get 6% interest. He says he knows a man who gets as high as 7% on a sum of money.

Yes, it is possible in some places to get even 8% interest on an investment, but we must be very careful in such matters. A low rate of interest usually means a safer investment than

a high rate. We should be pretty careful to investigate any offers of high rates of interest that we may get. A man should go for advice to some reputable banker in whom he has confidence. Unfortunately many fraudulent investments are offered to the public, while others are so risky as to be undesirable. Each one of us should learn something about distinguishing between good and bad investments. No one should ever invest a single dollar without careful investigation, preferably confirmed by the judgment of some banker or broker who is known to be responsible.

PUPIL. Is it better to buy stocks or bonds?

TEACHER. That is a good question to ask, but in order to give a satisfactory answer we shall have to take more time than we have left today. Tomorrow we shall take up certain important matters of banking practice, and in due time we shall take up the matter of good and bad investments. In that connection we shall consider stocks and bonds and try to find what the difference between them is and to decide which of the two is usually the better.

QUESTIONS AND TOPICS FOR DISCUSSION

1. In what ways may a young or inexperienced teacher profit by seeing a lesson taught or by reading a report of such a lesson?

2. Write a plan for teaching a lesson on one of the following topics, using the scheme presented in this chapter as a guide:

a. Various types of triangles.
b. How to find the area of a triangle.
c. A simple problem in the geometry of position.
d. The arithmetic of banking.
e. How to introduce the notion of a tangent.
f. How to prove an original exercise in geometry.
g. The first lesson in demonstrative geometry.
h. An appreciation lesson in algebra.

3. Under what conditions would a supervisor be justified in giving an inexperienced teacher a model lesson plan and then judging him by the way he used it in teaching rather than requiring him first to write his own plan? Give the reasons for your answer.

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CHAPTER XI

THE PLACE OF TESTS IN THE TEACHING OF MATHEMATICS

1. Introduction

Purpose of the Chapter. It is the purpose of this chapter to present a brief review of the various types of tests in mathematics that have been used in this country in recent years, to point out the advantages and the disadvantages of each, to discuss the progress that has been made in such tests, and to suggest better plans for their use in the future.

New-Type Tests. It is only within the last generation that any attempt has been made to bring about genuine reform in the method of measuring the effectiveness of instruction in secondary schools. In fact, the testing movement in its present form hardly antedates the World War. The main object of the chapter will be the discussion of a new type of testing program, — one in which the emphasis is placed upon a method of procedure rather than upon any single test itself. Moreover, whatever types of tests are recommended, each should be looked upon as a means of helping to improve the quality and effectiveness of instruction.

2. Broad Classification of Tests

Three Kinds of Tests. For the purpose of our discussion tests may be broadly divided into three classes. These are determined by the purposes which each is intended to serve, and may be described as prognostic tests, achievement tests, and diagnostic tests. We shall later have occasion to divide the last two into further classes for the purposes we have in mind.

Prognostic Tests. Prognostic tests are those which are usually given early in the pupil's career, and are intended to measure

his innate ability to do any specific kind of work. By their use a teacher is enabled to predict the probable success of a pupil in his later studies. For example, Professor Rogers's prognostic tests [1] were designed to measure the ability of each member of a class to succeed in the study of secondary mathematics.

Why Prognostic Tests were Developed. Prognostic tests in mathematics were developed to meet the need for some kind of instrument by which to measure a pupil's capacity to profit by the study of some special subject, so that he might be more intelligently advised with respect to his subsequent work. Since previous tests of this type, except those relating to geometry, dealt only with the more mechanical phases of the work, mathematicians objected to them on the ground that there is little in common between habits of symbol manipulation, which were emphasized by nearly all the early tests, and the more fundamental processes of reasoning, which are characteristic of higher mathematical study. In general, they maintained that the intensive tests in special fields like arithmetic, algebra, and geometry failed of their purpose for the reason that the topics tested are rarely applied by themselves. It should be observed, however, that everything depends upon the purpose which the maker of the test intends it to serve. If it is not intended to be prognostic, there may be some other justification for its use.

No one seems to have claimed that prognostic tests can be, or should be, the sole basis for prediction. Nevertheless, they constitute an important step in the direction of trying to discover those pupils who give the most promise in the field of mathematics. If such tests can be of assistance in discovering individual differences in mathematical ability, they may become useful instruments for prognosis in our schools. It should be insisted, however, that they must not be made the sole basis for prediction, classification, or guidance, because such absolute dependence would overlook other important factors. Moreover, there is no prognostic test available today whose use will enable

[1] A. L. Rogers, Tests of Mathematical Ability and Their Prognostic Value, Teachers College Contributions to Education, No. 89.

us to predict success in mathematics to a higher degree of certainty than that reached by any good test which measures abstract intelligence.

Disadvantages of Prognostic Tests. The prognostic test at its best achieves quickly and with more satisfactory results that which the schools have heretofore discovered after a loss of valuable time; at its worst it leads to a determinism that is more dangerous than the extreme form of Calvinism, which left each individual absolutely without hope. On the whole, the tests have achieved a great and well-deserved success, and this success will be much more apparent when a new generation comes forward to correct the errors of the present one.

Danger in Overuse of Prognostic Tests. The purpose in the minds of those who have tried to devise satisfactory prognostic tests is a laudable one. If properly planned, they served to test the general alertness of a pupil, his general ability to reason, and his interest in the specific things with which he has come in contact. However, the trouble lies in the fact that we cannot predict with any high degree of certainty what the pupil will do in a subject which he has had no opportunity to study. We can merely say that, other things being equal, the pupil is generally quick intellectually and, in so far as he has had experience with a certain subject, shows this or that kind of familiarity with it and a certain amount of regard for it.

Moreover we have had too many examples of supposed dullards who later became famous mathematicians to justify us in trying to reach hard-and-fast conclusions on the basis of any single test. A prominent mathematician relates that he did not wake up to his possibilities until he proved the theorem about the sum of the interior angles of a triangle. Another admits that he "never can figure and have it come out right." Some have developed late in life as was true of Sir Francis Galton, — not a pure mathematician but one who successfully applied mathematics in a very useful field. Much of his great work was not done until he had passed middle life.

Finally, there is perhaps no one, unless it be the highly-trained specialist in psychology, whom we should trust with the

task of predicting with any great certainty of correctness the possible futures for all the boys and girls in our American schools.

Future Value of Prognostic Tests. If these tests could be made in such a way as to become a reliable help in prognosis, we should all favor their wider use in selecting those pupils who are most likely to profit by taking advanced work in mathematics. In that case we might, and probably should, require mathematics only through the ninth year (the third year of the junior high school), the subsequent courses being offered only to those who are capable of a much higher grade of work than is now possible. Such prognostic tests would then be the means of doing justice to a large number of pupils who are forced to study mathematics against their will and in many cases without much gain. At least the tests could be used to advantage in determining the kind of mathematics that would be most valuable to a given pupil. If we had in our classes in mathematics of the tenth, eleventh, and twelfth years only those pupils who liked the subject and were able to study it with undoubted profit, the joy that would come to teachers and pupils alike would more than offset any feeling respecting the loss in numbers that would result from the adoption of such a program.

Achievement Tests. Whereas prognostic tests are concerned with ascertaining what a pupil is able to learn, it is the function of achievement tests to determine what he actually has learned. Such tests are intended to measure the progress which a pupil or a class is making in a given topic or course. They also measure indirectly the pupil's innate ability, since upon this is dependent whatever progress he may make. These tests, however, are so affected by classroom influences that they cannot be accepted as reliable instruments for the accurate measurement of the pupil's capacity to learn, although in general they correlate rather highly with intelligence.

Diagnostic Tests. Both prognostic and achievement tests may rightly be considered diagnostic in nature, although we have thus far made little use of diagnosis in the former type. Furthermore, an achievement test may be made diagnostic even

though not so intended, and an achievement test may have diagnostic value and still not be used for purposes of diagnosis. At any rate it is the purpose of the diagnostic test to find out, if possible, why any special pupil is unable to succeed, and to reveal strengths as well as weaknesses.

Value of Diagnostic Tests. Because the main purpose of diagnosis is to guide the teacher in remedial instruction, — that is, in remedying defects which the tests reveal, — it is proposed to emphasize later in the chapter the great importance of diagnostic tests as the best instrument for aiding the teacher in improving instruction. Without some plan of discovering the particular defects of a pupil or a class, the work of the teacher is likely to be more or less futile.

If the proper kinds of tests are given, the progress made by pupils can be measured, and the norms of accomplishment thus established can be used as a guide in discovering the educational needs of future classes. In addition to this the pupils assist the teacher greatly by locating not only their particular difficulties but the causes of these difficulties as well. The assistance in diagnosis rendered by the pupils' introspection should not be overlooked, because a pupil will often find the cause of a difficulty more quickly than the teacher.

3. Types of Achievement Tests

Possible Types. Two types of achievement tests have had a rather wide use in this country. The first type includes the tests set by the teacher alone or in cooperation with others who are more or less responsible for the existing course of study; the second includes the "extramural" tests set by examiners who have little or no direct contact with the classroom. Illustrations of the latter type are the ordinary College Entrance Board examinations, certain well-known state examinations such as the New York State Regents examinations, and the so-called standardized tests or scales.

It is not the intention to minimize the importance of either of the general types mentioned above, especially when tests are so

designed as to serve some useful purpose, but rather to encourage a wider use of certain special types and a more intelligent use of others.

All the tests referred to above, except the standardized tests or scales, belong to the class that is ordinarily known as "essay-type" examinations.

Certain modern writers recognize three types of achievement tests as follows:

1. The traditional "essay-type" examination.
2. Standardized tests or scales.
3. The "new-type" objective examination.[1]

The subsequent discussion will cover all these various types.

Tests set by the Teacher. A volume of considerable size could be written upon the subject of tests set by the teacher, but space will permit of only a brief statement of a few of their advantages and disadvantages. It is well known that many teachers are not qualified to make the careful analysis of the course which is necessary in determining the abilities which are fundamental and which can or should be measured. As a result certain teachers include in their final examinations a meager sampling of the large range of abilities, and regard these few questions as an instrument for measuring the entire field. Such a procedure is unscientific and depends too much upon the judgment of one person.

It is not so easy to make a good examination as many teachers seem to think. It takes a great deal of time, energy, and thought to construct a suitable test; but there is probably no part of our work today from which more important results would follow than from taking greater interest and care in the making of tests. We have known for a long time that our examinations have been inadequate to meet our needs, but little has been done to remedy the defect.

Marking Practices are Subjective. With many teachers the marking of test papers is highly subjective; that is, the mark given depends to a large extent upon the person who does the marking. Furthermore, the measures obtained by the teacher

[1] For an explanation of the term "objective" see page 343.

from many of the ordinary "essay-type" examinations are not truly diagnostic; that is, they do not point out the particular details upon which the pupil is either weak or strong. It is thus clear that if specific measures for diagnosis are desired, the ordinary examination will not prove entirely satisfactory. Instead of continuing the practice of doing injustice to thousands of pupils every year because our tests are misused, we must in some way or other develop a new marking system which will enable us to rate their achievements in relation to their ability.

Teachers best Qualified to do the Testing. In spite of the frequent inadequacy and inaccuracy of teachers' judgments, both in setting good examinations and in marking them fairly, it should be more generally realized that these same teachers are in the long run the people best qualified to do the task. If they do not already know, they can learn not only how to make objective tests that will have both measuring and diagnostic value, but also how to use them intelligently. This ability to use the test will increase in proportion to the progress they make in understanding scientific methods of measurement. Instead of constantly reminding teachers of their failures, without making suggestions for improvement, we should direct attention rather to the traditional method of marking, the continued presence of which in the system forces its use upon teachers to the exclusion of a better method. School marks have long been used to measure a multitude of things which have not been and cannot be measured with any degree of precision. By a proper development and use of objective tests we may be able to work out a better marking system for the future.

Extramural Examinations. During the past half century there have been many discussions regarding the examinations set by state and college authorities. Some progress has been made in the content of these examinations and in the method of conducting them, but much remains to be done before they can be said to have reached their highest development even for the purpose for which they are designed. New York State has dictated Regents examinations, generally good for a poor teacher and generally bad for a good one. The College Entrance Ex-

amination Board has also, and naturally, dictated what should be taught in mathematics, and has recently made a long step in advance by a series of improvements. Each of these cases of dictation has in the past contributed powerfully to making mathematics stagnant, and each has been potent in keeping it on a dead level of traditional mechanism.

Everybody will readily believe that those who formulate the tests referred to above are often better able to judge what is fundamental in a course than are most classroom teachers of mathematics. In fact, the recent College Entrance Examination Board syllabuses reflect more modern types of mathematics courses than those which a great many teachers are now using.[1] The trouble is not so much with the original purpose of such examinations as with their imperfect preparation, use, and interpretation. In many cases these examinations measure abilities which have not been considered in the previous instruction given by many of the teachers whose pupils have to take them, and consequently the results secured cannot be called fair measures of the teachers' effectiveness. Nevertheless it is true that the degrees of efficiency of teachers and of classes in the same or different schools have been compared on the basis of the relative scores made on such examinations. As a result all sorts of methods are resorted to by certain teachers whose sole aim is to get their pupils through these examinations with a passing grade. Such practices, it need hardly be said, do not represent a desirable type of modern education. Since these examinations, as now regulated and administered, have not given satisfactory results, the question arises as to whether or not it would be a wise plan to replace them by some form of intelligence test or prognostic test that could be used as a basis for determining those pupils who are able to undertake higher mathematical work.

College Entrance Board Examinations. Some of the shortcomings of the College Entrance Board examinations in algebra and geometry have been pointed out by Professor B. D. Wood.

[1] Report of the Secretary of the College Entrance Examination Board (1921), pp. 1–4; (1922), pp. 18, 19; (1923), pp. 1–3, 7–9.

Using two examinations that were set in June, 1921, — one in algebra, known as "Mathematics A," and one in geometry, known as "Mathematics C," — he finds that "the reliability of the examination itself in algebra is unsatisfactory, and in geometry entirely unacceptable." [1] If the instances which he cites are representative, he seems to have established a case against such examinations, even though a satisfactory remedy is not suggested.

Attitude of the General Public. The mere fact that there has been so much discussion of late about the value of "extramural" examinations [2] indicates that many thoughtful people are not yet satisfied with the present status of such tests.[3]

In a recent address to the friends and alumni of a prominent private school for boys, President Angell said, in part:

I have no quarrel with such examinations when wisely administered. . . . May I say that until a relatively recent time the colleges attempted to dictate in a good deal of detail just what the school should teach the boys. This resulted in occasioning embarrassment to schools teaching boys for colleges where requirements varied, and took away from the schools some measure of that responsibility for their own curricula which personally I believe to be indispensable to their best intellectual and educational development. In recent years this situation has greatly improved, both because the colleges have made their requirements much more flexible and because many schools have developed a more independent attitude. I believe that the endowed school gains greatly by developing its own individuality and avoiding a too stereotyped pattern.

In the second place, with the best of intentions to "teach the subject," many of these schools have the freedom and spontaneity of their procedure appreciably impaired by the compelling necessity so to train their boys that the latter may be substantially certain of passing the college entrance examination. The parents who send their sons to be prepared to enter a special college are naturally dis-

[1] B. D. Wood, The Reliability and Difficulty of the College Entrance Examinations in Algebra and in Geometry, p. 14. Published by the College Entrance Examination Board, 1921.

[2] M. Barnes, "Procrustes Redivivus," *Atlantic Monthly* (July, 1925), p. 83.

[3] W. M. Proctor, "The High School's Interest in Methods of Selecting Students for College Admission," *School and Society* (October 10, 1925), Vol. XXII, pp. 441–448.

appointed if the school fails at this point. It is inevitable that under these conditions the minds of the boys and the teachers alike should be somewhat unduly fixed on the mere passing of the dreaded examinations. With the much improved type of examinations now set by the College Entrance Examination Board, this source of anxiety should be reduced to a minimum, but that the work of certain of the schools is still disturbingly affected by this consideration I have substantial reason to believe.

The cure is not in the abolishment of proper examinations but in a recognition by schools and parents alike that the best teaching can only be done when the examination, as such, is not the controlling objective.

Attitude of Mathematics Teachers. That certain teachers of mathematics are not satisfied with College Entrance Board examinations in mathematics is also evidenced by a report of the Committee on Examinations of the Association of Mathematics Teachers of New Jersey.[1] In this report it is charged that the present College Entrance Board examinations in mathematics do not give the largest possible measure of specific preparation, — first, because the form of the examinations is at fault as a measuring instrument; secondly, because the topics chosen are not weighted in accordance with their value as a preparation for advanced study; and, finally, because the present "comprehensive" examinations are not what their name suggests, that is, they are not composites of examinations which, taken individually, seek to test important topics or abilities.

The Remedy for the Situation. It is therefore evident that these examinations do not meet the requirements as measuring instruments. They tend to measure less of mathematical achievement and more of general intelligence. Furthermore, such examinations cannot be given to all pupils except at the expense of certain groups. In fact, it is easy to conceive of a situation in which one group whose preparation had been of a higher type than that of another would make a showing distinctly inferior to that of the second group simply because the examination was not based upon material to which the first group had devoted an equal amount of attention.

[1] "Preliminary Report," *The Mathematics Teacher* (October, 1924), p. 374.

Since the College Entrance Board examinations, State Regents examinations, and the like are not devices which measure adequately a pupil's ability, why should they not be given their proper place and weight in our educational scheme? Better still, as suggested above, if their only purpose is to secure the most capable pupils for college work, why cannot examining boards be induced to give intelligence tests instead of the traditional examinations? If that were done, all high-school pupils could be given that type of mathematics which would be most useful to them whether they went to college or not. The only reason for not doing it would seem to be that it has never been done, or that the newer type of test has not been sufficiently perfected in, say, a subject like demonstrative geometry.

Standardized Tests. The so-called "standardized tests" have recently come into rather wide use in this country, but their introduction has been accompanied in many places by grave misuse. Although they are intended to be scientific, they have often been quite the reverse. Sometimes those in charge of the testing program are so unwise as to judge the success or failure of a teacher solely by the outcome of such tests. In some cases invidious comparisons made on the basis of test results have not been checked by a careful study of the methods of teaching employed. In still other cases certain standardized algebra tests have been regarded as entirely reliable, and it has been assumed that they could be used to measure every feature of the teaching of that particular subject.

The Trouble with Standardized Tests. A great deal of the traditional algebra which has been questioned by many teachers and textbook writers as contrary to modern objectives has been replaced by more valuable material. It is an interesting fact, however, that standardized tests have failed to include the latter type of material and thus have made difficult the general acceptance of some of the more modern algebra courses. The values claimed for standardized tests because of their carefully selected content have not always been evident. The complaint is not so much that the tests are solely mechanical, involving only a minimum of intellectual processes, — a fault

that is probably inevitable in the present stage of development, but which is being successfully removed in some of the arithmetic tests; it is also that the material required for testing the mechanical processes is often such as should play only a minor rôle, if any, in the education of the average citizen. The tests represent generally a dead level of dull grind, offering to the teacher only this ideal of an algebra course. He may escape from the curriculum, making his own course; he may and should select from the textbook that which he needs for carrying out his plan; but he cannot escape from teaching those things that are required by outside examinations, whether they be set by boards of regents, or by local educational authorities.

A careful study of many of the standardized tests shows that they have been prepared by people who select material with little regard to the proper objectives to be obtained and who seem to be ignorant of the plans for the reorganization of mathematics. Moreover, there exist many desirable objectives which were overlooked when they were made. Statistically the tests themselves may be perfect; but when a task is introduced into a test merely because it represents a certain degree of difficulty, it simply tends to prejudice all persons of common sense against the whole movement.

Improvement of Tests. Teachers always have measured and always will measure their pupils in some way or other, but the science of measurement will not reach its maximum of importance until the teachers and the makers of tests establish a partnership. Each may then hope to develop scientific methods of approach, and only then will the teachers themselves have an abiding interest in the outcome of their work.

Greatest Use of Standardized Tests in Mathematics. The greatest use of standardized tests has been made in arithmetic. This has been due to the fact that the material lends itself readily to standardization. Even in this field the tendency today is away from general national standardization and toward practice exercises and diagnosis of individual cases. This has been due to a realization of the importance of giving to the pupils their standing based upon some definite scale of per-

formance related to their own class rather than trying to place them with reference to a norm based on the performance of some outside group, — a practice due entirely to the influence of the recent testing movement.

Chief Values Obtained from Standardized Tests. The chief values obtained from the use of standardized tests in mathematics may be summarized as follows:

1. Such tests have brought out more clearly the problem arising from individual differences in ability among pupils.

2. They have shown us that a great deal of the traditional material is too difficult for most pupils and therefore should not be taught.

3. In some cases they have also shown that certain material was easier than had been expected and that it can be mastered by a large majority of pupils.

4. They have made it possible for the teacher to stop drilling certain pupils beyond the stage of diminishing returns.

5. They have made it possible to develop certain standards of achievement which are clearly defined and which can be assigned to varying levels of intelligence. As we shall see, however, this value has occasionally been overrated.

6. They have contributed to the development of more objective methods of testing.

7. They have, when intelligently used, stimulated pupils to renewed effort in trying to reach certain standards of perfection, or at any rate to improve upon their own previous records.

All the above outcomes have been worth while, but, as we have previously pointed out, it is equally true that standardized tests do not lend themselves readily to some of the more important needs which an ideal testing program would meet.

Professor Upton[1] has given a thorough discussion of the influence of standardized tests on the curriculum in arithmetic. One of the authors of this book has made certain criticisms of algebra tests,[2] and the other has offered suggestions for making

[1] C. B. Upton, "The Influence of Standardized Tests on the Curriculum in Arithmetic," *Teachers College Record* (April, 1925), Vol. XXVI, p. 627.

[2] D. E. Smith, "On Improving Algebra Tests," *Teachers College Record* (March, 1923), Vol. XXIV, pp. 87, 88.

a careful diagnostic study[1] of some of the teaching problems in high-school mathematics. At the present time it would seem that we should be more interested in determining clearly the purposes in view in the teaching of mathematics, the content best fitted to help us realize these purposes, and the kind of tests that shall afford a check upon our results. This does not mean that no measuring should be done in the meantime, but rather that our methods of measuring should be improved before we seek to increase the use of standardized tests.

In all fairness to these tests, however, it should be said that they have gone beyond what Professor Woody calls the "curiosity" stage and "the stage in which the predominant idea was the use of the tests for determining existing levels of achievement," and, in some respects at least, have approached the third stage, "in which the predominant idea is the utilization of tests as a means for the improvement of instruction." They have been helpful in this third stage, however, only as they furnish facts concerning certain "levels of efficiency" reached by pupils and thus "contribute to the evaluation and diagnosis of the efficiency of instruction."

Use to be made of Standardized Tests. Where standardized tests are valid and reliable instruments, they may be profitably used for purposes of "general-survey diagnosis," and even in some cases for class and individual diagnosis; but this work must be based more and more upon the cooperation of all concerned, from the superintendent down to the pupils themselves.

National Committee on Standardized Tests. In Chapter XIII of the Report of the National Committee on Mathematical Requirements Professor Upton gave a careful and complete discussion of standardized tests in mathematics for secondary schools. His report not only included a description and discussion of the standardized tests in use since 1914, but also gave illustrations of special tests in arithmetic, algebra, and geometry, and of those concerned with the measuring of general mathemat-

[1] W. D. Reeve, A Diagnostic Study of the Teaching Problems in High School Mathematics. Ginn and Company, 1926.

ical ability. Teachers who are interested in knowing more about the nature and content of such tests will find this chapter of the National Committee Report a valuable source of information and help.

Use of Standardized Tests not Obligatory. While standardized tests have been of real service in the ways that have been pointed out and therefore are to be commended for serving the purposes for which they were intended, nevertheless there are reasons at present for giving them less prominence in educational discussions and for turning our attention to a testing program that seems to offer an opportunity to obtain more important results.

What we may expect in Future Tests. It seems reasonable to expect that we shall in due time have tests that are prepared by those who know not only the underlying principles of these devices but also the true objectives in the teaching of mathematics. These tests can be made in such forms that they may be used not only for both general and particular diagnosis but also as aids in improving instruction. When this is done, there will follow the establishment of norms of attainment which can be generally used. We should realize, however, that a teacher is not always justified in feeling satisfied with the results when pupils reach a certain arbitrary norm of achievement. This would amount to the standardizing of mediocrity. To do this is to overlook the fact that the standard norms may be raised by lifting the general level of achievement through better methods of teaching.

It is also reasonable to expect that we might reach our educational objectives without the type of test mentioned above; but it is hardly conceivable that we shall be able to succeed as we wish unless we develop, for mass instruction, a type of teaching that is based upon the specific needs of individual classes and pupils. This sort of instruction should be remedial and must be based upon specific diagnostic measures of pupils' achievements.

It is well known that there are differences of opinion with respect to what it is important to teach in a given course in

mathematics. Until we have a more carefully considered collection of tests, it is futile to expect them to serve any very useful purpose. When that time comes, many teachers will consider them no less important than their textbooks. They will then use tests both as teaching devices and as measuring instruments in all their work. This will be a great improvement over some of our present practices, where teachers who feel the pressure of tests set from without fall into the habit of making such tests their only textbooks.

4. The Place of Tests in Curriculum Construction

Steps involved in Curriculum Construction. Obviously, the most important, or at least the most discussed problem before us today is that of curriculum construction. The task of reorganizing mathematics so as to provide a desirable body of material for the junior and senior high school involves at least four main steps.

The first step is to agree upon a list of desirable objectives to be attained in the teaching of mathematics. The second step is to determine the nature and the extent of the subject matter which will best enable teachers to realize the chosen objectives. The third step is to develop the best methods of teaching the selected subject matter. This cannot be done without some kind of testing program whereby the teacher is able to discover to what extent his methods are paying dividends. No matter how desirable the content may seem to be and how well his methods are perfected, it may be that the material is too difficult for the pupils in a given year. Moreover, it may also be true that even though it is possible for children to learn certain things in mathematics at a given age, the time required for learning them is too great to justify their inclusion in the course of study for that year. We must therefore have a fourth step, which is a testing program that will enable us to see how well the pupils are learning the things we have been trying to teach. This last step necessarily involves a more careful analysis than heretofore of how pupils learn most efficiently and easily.

Guiding Principles underlying Good Tests. The following guiding principles underlie the construction of a good test in mathematics:

1. *A test should attempt to increase the pupil's ability to master the subject matter that has been presented to him.* This means that the teacher must discern clearly the objectives[1] in the topic or course and must build his examination so as to measure the extent to which these objectives have been realized. Such a procedure measures the progress made by a pupil or a class. This is frequently taken as the first aim of an examination. A test which has no reference to what has already been taught cannot meet this requirement. If pupils who have the native capacity to learn a certain thing fail to do so, there is an obvious opportunity for remedial work. On the other hand, there is little to be expected from or gained by remedial work in the case of pupils who have done as well as can be expected considering their innate capacity. If both the diagnosis and the subsequent remedial work suggested by the tests are to be of value, the teacher must have at hand specific measurements of the pupil's achievement. Such measurements can be fairly made only in a field in which the pupil has been working.

Moreover, tests that contain material that has not been previously taught cannot have much diagnostic value. In such a case it is impossible to tell whether poor results mean that the content of the course was too difficult, that the teaching was not well done, or that the material was entirely unknown to the pupils.

2. *Every test should emphasize those parts of the subject matter which are fundamental and to which the pupils have directed the most attention.* Nothing should receive attention that is not worth perpetuating in the course. This means that every test should contain a thorough sampling of the fundamental ideas of a topic or a course for the complete mastery of which the pupil is held responsible. In other words the test must be comprehensive. This has never been true of the traditional "essay-type" examination.

[1] W. D. Reeve, "Objectives in the Teaching of Mathematics," *The Mathematics Teacher* (November, 1925), Vol. XVIII, pp. 385–405.

If the two preceding principles are adopted the test will be ranked as valid or as having "validity," — that property of a test which is supposed to represent the degree to which a test measures what it is intended to measure.

3. *The scoring of every test should be made as objective as possible.* In this way different teachers employing the same test in measuring the same abilities may obtain exactly the same results. This means that the personal factor must be largely removed from the scoring of results and that teachers in framing the papers should be careful about the mechanics of the tests. If this principle is kept in mind, not only will the response of the pupils be more uniform, but the marking practices of those who score the papers will be less variable. These ends are achieved by making the number of items in the test as large as possible and by requiring for each of these items a definite response to which all persons scoring the test papers will readily agree. The failure to do this is responsible for the chief defect of many "essay-type" examinations.

4. *Every test should be reliable.* Reliability is that property of a test which determines the degree to which a test "measures what it really does measure." A test may measure what it is intended to measure, but it may do it very unreliably. The determination of the reliability of a test is, however, a more technical matter and need not be further discussed here. It is discussed fully in some of the recent books on examinations.

5. *Every test should be so constructed that it is almost self-administering.* It should be made so that it can be easily given and scored by an intelligent person who may or may not have had much mathematical training. In the past this has not often been done. The motto should be: "Hard to make but easy to give and score."

6. *Every test should make it possible to set some sort of standard of achievement for a pupil.* Such a standard may match him against his own group or against his own record.

Practice Tests. In any testing program the teacher should introduce practice tests both as diagnostic instruments and as teaching devices. In fact, such tests are so important that they

are fast coming to be an integral part of the newer textbooks. They serve a purpose in testing specific objectives which have to be covered in a very short period of time, — a function unknown to most standardized tests. Whether timed tests are to be chosen is a matter for the teacher to determine. Certainly no undue emphasis should be placed upon speed at the expense of accuracy.

Individual Instruction Important. It is clear that at best much time has been lost in trying to make an economical combination of mass instruction and individual instruction. As between the two, it seems evident that individual instruction should receive the greater emphasis. One great value of the practice tests lies in the fact that they enable the teacher to discover quickly the pupils who are in need of help, and to keep from wearying the brilliant pupils with work that they do not need. Furthermore, they help the pupil to measure his own progress in any given topic, and to be more intelligent in calling upon the teacher for assistance when this progress is not satisfactory. There is much truth in the statement that "the child learns the responses which he makes." For the pupils who need to have certain skills more highly developed, practice tests are invaluable, while at the same time they enable the teacher to excuse others from practice work which might easily become repulsive, even though overlearning is better than underlearning.

Value of Practice Tests. A large number of practice tests can and should be made for use in connection with the various topics. For example, the one on page 345 is a timed practice test on the first type of equation discussed on page 347. Such practice tests will save the teachers a great deal of time in the preparation of material for use in the classroom provided the textbook writers make the proper analysis of the unit skills to be developed and then include in their books enough exercises to test the pupil's mastery of such skills. If such tests are not included in the textbook used by the teacher, he should carefully analyze the skills that he wishes the pupil to master. He should then make tests to fit that analysis and use them both as teaching and as testing devices.

The question of time limits in tests of this nature is relatively unimportant; in fact, experience has shown that the chief value of such limits is to encourage reasonably rapid work on the part of the pupils. The individual differences of pupils with respect to speed are so great that any attempt to insist upon a standard rate is certain to be futile.

TEST NO. 19

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

FIRST TYPE OF EQUATION

Time: 5 min.

In each of the following cases write the value of the specified letter after the sign of equality:

1. $2x = 4;\ x =$
2. $3x = 9;\ x =$
3. $8 = 4x;\ x =$
4. $5x = 5;\ x =$
5. $6x = 6;\ x =$
6. $2n = 6;\ n =$
7. $12 = 2n;\ n =$
8. $3a = 15;\ a =$
9. $4b = 16;\ b =$
10. $5p = 35;\ p =$
11. $40 = 4x;\ x =$
12. $7y = 49;\ y =$
13. $6q = 66;\ q =$
14. $8x = 72;\ x =$
15. $9x = 54;\ x =$
16. $3a = 36;\ a =$
17. $5b = 70;\ b =$
18. $6y = 54;\ y =$
19. $7z = 63;\ z =$
20. $8c = 64;\ c =$
21. $1 = 2x;\ x =$
22. $3x = 2;\ x =$
23. $3 = 4a;\ a =$
24. $5a = 5;\ a =$
25. $12 = 6a;\ a =$
26. $7a = 10;\ a =$
27. $4 = 8y;\ y =$
28. $0.8y = 4;\ y =$
29. $1.8y = 18;\ y =$
30. $3 = 9y;\ y =$
31. $0.9y = 9;\ y =$
32. $1.8 = 0.9y;\ y =$
33. $15 = 2z;\ z =$
34. $2z = 21;\ z =$
35. $4 = 5n;\ n =$
36. $0.3n = 6;\ n =$
37. $4b = 5;\ b =$
38. $1.4b = 2.8;\ b =$
39. $13 = 6x;\ x =$
40. $1.6x = 16;\ x =$
41. $10a = 10;\ a =$
42. $10a = 5;\ a =$
43. $10a = 8;\ a =$
44. $20 = 10a;\ a =$
45. $25 = 10a;\ a =$
46. $12x = 24;\ x =$
47. $1.2x = 2.4;\ x =$
48. $1.2x = 24;\ x =$
49. $12x = 36;\ x =$
50. $1.2 = 1.2x;\ x =$
51. $3.6 = 1.2x;\ x =$
52. $20n = 40;\ n =$
53. $10 = 20n;\ n =$
54. $5 = 20n;\ n =$
55. $25 = 20n;\ n =$
56. $25y = 50;\ y =$
57. $50y = 25;\ y =$
58. $50y = 5;\ y =$
59. $36x = 36;\ x =$
60. $36x = 3.6;\ x =$

Composite Tests. At frequent intervals it will be necessary to construct composite tests that contain certain representative exercises from preceding practice tests, as shown in the model test given below. Such tests should not be given until after the various skills included in them have been developed separately and for a long enough time to insure a high degree of mastery.

TEST NO. 27

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

FIRST FOUR TYPES OF EQUATIONS

Time: 8 min.

In each of the following cases write the value of the specified letter after the sign of equality:

1. $2x = 4$; $x =$
2. $\frac{1}{2}x = 5$; $x =$
3. $x + 7 = 13$; $x =$
4. $x - 8 = 5$; $x =$
5. $2x = 6$; $x =$
6. $2x + 1 = 7$; $x =$
7. $2x - 1 = 5$; $x =$
8. $\frac{1}{2}x = 4$; $x =$
9. $\frac{1}{2}x + 1 = 5$; $x =$
10. $\frac{1}{2}x - 1 = 3$; $x =$
11. $3x + 1 = 7$; $x =$
12. $4x - 1 = 3$; $x =$
13. $4x - 3 = 1$; $x =$
14. $7 = 4x - 1$; $x =$
15. $1 = 4x - 7$; $x =$
16. $5x - 2 = 3$; $x =$
17. $5x - 3 = 2$; $x =$
18. $\frac{x}{2} = 1$; $x =$
19. $\frac{x}{2} + 1 = 2$; $x =$
20. $\frac{x}{5} + 3 = 7$; $x =$
21. $9x - 2 = 7$; $x =$
22. $9x - 7 = 2$; $x =$
23. $2x - 9 = 7$; $x =$
24. $3x - 0.9 = 9$; $x =$
25. $7x + 2 = 9$; $x =$
26. $2x + 7 = 9$; $x =$
27. $9x + 1 = 10$; $x =$
28. $2x - 1 = 17$; $x =$
29. $2x - 0.1 = 1.7$; $x =$
30. $20 = 19x + 1$; $x =$
31. $19 = 20x - 1$; $x =$
32. $8 = 10x - 2$; $x =$
33. $8 = 2x - 10$; $x =$
34. $10 = 2x - 8$; $x =$
35. $8x - 2 = 14$; $x =$
36. $6x + 0.2 = 14$; $x =$
37. $2x - \frac{2}{3} = 6$; $x =$
38. $\frac{x}{10} - 1 = 1$; $x =$
39. $3 = \frac{x}{10} + 1$; $x =$
40. $\frac{x}{20} + 1 = 2$; $x =$
41. $\frac{1}{2}x + 1 = 2$; $x =$
42. $0.5x + 1 = 3$; $x =$
43. $\frac{1}{4}x - 1 = 1$; $x =$
44. $0.25x + 1 = 4$; $x =$
45. $\frac{1}{5}x - 2 = 1$; $x =$
46. $\frac{1}{5}x - 0.2 = 1$; $x =$
47. $\frac{1}{5}x + 2 = 3$; $x =$
48. $0.2x + 3 = 5$; $x =$
49. $5 = \frac{1}{5}x - 3$; $x =$
50. $5 = \frac{1}{5}x - 0.5$; $x =$
51. $2 = \frac{1}{6}x + 1$; $x =$
52. $0 = \frac{1}{6}x - 6$; $x =$
53. $6 = \frac{1}{6}x + 6$; $x =$
54. $7 = \frac{1}{6}x + 5$; $x =$
55. $\frac{2}{3}x + 1 = 1\frac{2}{3}$; $x =$
56. $\frac{2}{3}x + 1 = 3$; $x =$
57. $\frac{2}{3}x - 1 = 1$; $x =$
58. $\frac{2x}{3} - 1 = 3$; $x =$
59. $\frac{2x}{3} + 1 = 5$; $x =$
60. $5 = \frac{3x}{4} + 2$; $x =$

The Testing of Specific Objectives. The method of procedure in making such tests can best be understood by taking certain definite objectives, upon which it will be assumed we are agreed, and building up the practice tests and composite tests which seem to be necessary to ascertain to what extent our aims are being realized. Let us assume for the sake of illustration that we have decided upon the following specific objectives in teaching the formula:

1. *To develop certain rules of mathematics and to translate them into formulas.* This means that pupils should understand the meaning of the formula as a shorthand rule of mathematics. This rule should, in general, grow out of their experience. At any rate, so far as possible they should be told what the formula means. This is where algebra properly begins.

2. *To translate certain formulas into rules of mathematics.* This means that pupils must know how to use a formula when the need arises. Obviously they cannot do so unless they can translate the formula into a rule of procedure.

3. *To evaluate certain formulas.* This means that we are to find the values of certain letters when the values of the others are known. The formulas should be of a difficulty no greater than that found in the operations which the pupils have been taught or which they may be expected to understand.

The types of equations involved are as follows:

(1) $2w = 16$	(3) $p + 4 = 104$
(2) $\frac{1}{2}h = 4$	(4) $n - 4 = 7$

The need for solving an equation like $2w = 16$, or $16 = 2w$ arises in using a formula like $A = lw$. For example, if the area of a rectangle is 6 and the length is 2, the width is found by solving the equation $6 = 2w$. The necessity for solving the second type arises in using a formula like $A = \frac{1}{2}bh$. For example; if $A = 4$, $b = 1$, and the height is to be determined, we must solve the equation $\frac{1}{2}h = 4$, or $4 = \frac{1}{2}h$. In like manner, if in using the formula $A = p + i$ we know that $A = 104$ and $i = 4$, and need to find p, we must solve the equation $104 = p + 4$, or $p + 4 = 104$. Similarly, if in the formula $A - p = i$ we know

that $p = 100$ and $i = 5$, and wish to find A, we must solve the equation $A - 100 = 5$, or $5 = A - 100$.

4. *To derive one formula from another.* This means that the pupil must be able to solve a formula for one letter in terms of the other letters involved. Some writers refer to this as "changing the subject of the formula," but this phrase is not a good one to use in the classroom because it is apt to be confusing to the pupil and thus adds to his difficulty.

If the pupils have sufficient drill in solving the first four types of equations given above, they will be able to derive one formula from another. Thus, if they can solve such equations as $2w = 16$ for w, they will be able to solve the equation $lw = A$ for w, and so on. Care should be taken, however, to develop the work inductively.

5. *To represent by a graph certain formulas.* This involves the ability to make a table of values for a formula. The limit of difficulty in this work should be the Fahrenheit-centigrade formula $F = \frac{9}{5}C + 32$.

6. *To understand the idea of dependence of one quantity upon another.*

The tests that follow on pages 349–355 are samples of those which might be used in connection with the list of objectives given above.

It is not expected that these samples will cover every need. The rules to be translated and the formulas to be learned will vary widely according to local needs. Most of the rules given on page 349, however, will be included in every list that is taught. Similarly, in the selection of the formulas to be evaluated it will be necessary to consider the local needs and the abilities of the pupils. The formulas selected need not be the same ones that were previously translated into rules, but they should be formulas which can be simply stated and which do not involve too complicated computations in their evaluation. A mastery of the essential parts of algebra depends to a large extent upon the power acquired in relatively simple operations, and for this reason emphasis upon reasonably rapid work in a large number of simple cases is desirable.

TEST NO. 100

NAME.................... DATE....................

CHECKED BY .... RIGHTS ....

TRANSLATING STATEMENTS INTO FORMULAS

In the right-hand column write the formula which expresses the corresponding statement at the left:

STATEMENT	FORMULA
1. The area of a circle is π times the square of the radius.	1.
2. The area of a rectangle is the product of the length and width.	2.
3. The area of a triangle is half the product of the base and height.	3.
4. The area of a trapezoid is half the product of the height and the sum of the bases.	4.
5. The circumference of a circle is 2π times the radius.	5.
6. The volume of a rectangular solid is the product of the length, width, and height.	6.
7. The volume of a cylinder is the product of π, the height, and the square of the radius.	7.
8. The area of the lateral surface of a cylinder is the product of 2π, the radius, and the height.	8.
9. The volume of a prism is the area of the base times the height.	9.
10. Simple interest is the product of the principal, rate, and time.	10.
11. The volume of a cone is one third the product of π, the height, and the square of the radius.	11.
12. The volume of a pyramid is one third the product of the height and the area of the base.	12.
13. The distance traveled by a moving body is the product of the rate (or speed) and the time.	13.
14. The area of the surface of a sphere is four times the product of π and the square of the radius.	14.

The reverse test, the recognition of the meaning of formulas, or the translation of formulas into rules, can readily be made from the test given above. All that it is necessary to do is to write the formulas on the left-hand side of the page and leave sufficient space for the pupil to write the corresponding rules on the right-hand side.

TEST NO. 30

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

RECOGNIZING FORMULAS

Time: 5 min.

Match the columns below by writing the number of the corresponding description in the square in front of each formula in the right-hand column:

1. The formula for the area of a rectangle.	☐ $T = 6\,e^2$.
2. The formula for the circumference of a circle.	☐ $A = p(1 + rt)$.
3. The formula for the volume of a pyramid.	☐ $h = \frac{2\,A}{b}$.
4. The formula for the area of a parallelogram.	☐ $s = \sqrt{A}$.
5. The formula for the total length of all the edges of a cube.	☐ $V = \frac{1}{6}\,\pi d^3$.
6. The formula for the area of the total surface of a cube.	☐ $A = lw$.
7. The formula for the area of the surface of a sphere.	☐ $C = \pi d$.
8. The formula for the volume of a sphere.	☐ $A = \frac{1}{2}\,h(b + b')$.
9. The formula for the edge of a cube in terms of the volume.	☐ $S = 4\,\pi r^2$.
10. The formula for the amount of a sum invested at simple interest.	☐ $V = \frac{1}{3}\,Bh$.
11. The formula for the principal in terms of the interest, rate, and time.	☐ $r = \sqrt{\frac{A}{\pi}}$.
12. The formula for the side of a square in terms of the area.	☐ $A = bh$.
13. The formula for the area of a trapezoid.	☐ $A = \frac{1}{4}\,\pi d^2$.
14. The formula for the radius of a circle in terms of the area.	☐ $L = 12\,e$.
15. The formula for the area of a circle.	☐ $V = \frac{1}{3}\,\pi r^2 h$.
16. The formula for the altitude of a triangle in terms of the area and base.	☐ $p = \frac{i}{rt}$.
17. The formula for the volume of a cone.	☐ $e = \sqrt[3]{V}$.

In the above test both the formula and its description are given, and the pupil is required to match the two columns. In addition to the interest aroused by this type of test, the work is important in training the pupil to recognize at sight the important formulas of mathematics, — an ability which will be extremely useful in subsequent work.

TEST NO. 5

NAME .... DATE ....

CHECKED BY .... RIGHTS ....

EVALUATING FORMULAS

Time: 10 min.

Using the formula $A = lw$, which means that the area of a rectangle is the product of the length and the width, complete the following statements:

1. If $l = 4$ and $w = 3$, I find the product of 4 and ..... and thus know that $A =$

2. If $l = 3$ and $w = 4$, I .... 3 by 4 and find that $A =$

3. If $l = 6$ and $w = 3$, I find by multiplication that $A =$

4. If $l = 3\frac{1}{2}$ and $w = 6$, I multiply ..... by ..... and find that $A =$

5. If both l and w have the value 9, then $A =$

Using the above formula for the rectangle, fill in the column marked A in each of the following tables:

	l	w	A
6.	4	7	
7.	6	3	
8.	$5\frac{1}{2}$	8	
9.	8	$3\frac{1}{2}$	
10.	7	$2\frac{1}{2}$	

	l	w	A
11.	5	$3\frac{1}{4}$	
12.	4	$4\frac{3}{4}$	
13.	$5\frac{3}{4}$	16	
14.	$2\frac{1}{2}$	$3\frac{1}{2}$	
15.	$3\frac{1}{2}$	$4\frac{1}{4}$	

	l	w	A
16.	$5\frac{1}{2}$	$6\frac{3}{4}$	
17.	$6\frac{1}{4}$	$5\frac{1}{8}$	
18.	$6\frac{1}{2}$	$5\frac{1}{8}$	
19.	$7\frac{3}{4}$	$6\frac{3}{8}$	
20.	$5\frac{3}{4}$	$8\frac{5}{8}$	

Using the formula $d = rt$, which means that the distance traveled by an automobile is the product of the speed r (in miles per hour) and the number of hours t, complete the following statements:

21. If $r = 25$ and $t = 4$, I multiply mentally and find that $d =$

22. If $r = 22\frac{1}{2}$ and $t = 5$, I multiply ..... by ..... and find that $d =$

23. If $r = 28$ and $t = 3\frac{1}{2}$, I multiply ..... by ..... and find that $d =$ .....; that is, in $3\frac{1}{2}$ hr. an automobile will travel ..... mi. at ..... mi. per hour.

24. If $r = 32.4$ and $t = 2\frac{3}{4}$, I find that $d =$ .....; that is, the distance traveled by the automobile in ..... hr. is ..... mi.

In the practical applications of algebra the most frequent use that is made of the subject involves evaluation of some sort. Work of the kind given in the above test is also important in showing the pupils the wide range of cases to which a single formula can be applied and thus leading them to appreciate the great power of algebraic methods.

TEST NO. 8

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

EVALUATING FORMULAS

Time: 5 min.

In the third column of the following table insert the proper values of the specified letters:

	FORMULA	GIVEN VALUES	VALUE FOUND
1.	$A = lw$	$l = 18$ and $w = 9$	$A =$
2.	$A = bh$	$b = 5\frac{1}{4}$ and $h = 10\frac{1}{2}$	$A =$
3.	$A = \frac{1}{2}bh$	$b = 3\frac{1}{2}$ and $h = 10$	$A =$
4.	$T = nc$	$n = 12$ and $c = \$30$	$T =$
5.	$T = nc$	$n = 36$ and $c = \$0.35$	$T =$
6.	$A = \frac{1}{2}h(b + b')$	$h = 8$, $b = 6$, and $b' = 4$	$A =$
7.	$C = 2\pi r$	$\pi = \frac{22}{7}$ and $r = 21$	$C =$
8.	$C = 2\pi r$	$\pi = \frac{22}{7}$ and $r = 28$	$C =$
9.	$A = \pi r^2$	$\pi = \frac{22}{7}$ and $r = 14$	$A =$
10.	$V = lwh$	$l = 4$, $w = 3$, and $h = 2\frac{1}{3}$	$V =$
11.	$C = \pi d$	$\pi = \frac{22}{7}$ and $d = 14.7$	$C =$
12.	$A = \frac{1}{4}\pi d^2$	$\pi = \frac{22}{7}$ and $d = 7$	$A =$

After the rights have been checked, do over in the space below any exercise in which your result was incorrect, so that you can find the source of your error.

As has been previously stated, the help that the pupil can render in diagnosing his own difficulties should not be overlooked. In work like that given above, the pupil's errors will be largely due to carelessness, but the plan of having the pupil locate the source of his errors for himself is one that can well be followed more often than it is.

TEST NO. 31

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

DERIVATION OF FORMULAS

Time: 8 min.

In each of the following cases derive a formula for the letter specified:

1. $A = lw$; $w =$
2. $V = Bh$; $B =$
3. $T = nc$; $c =$
4. $i = prt$; $t =$
5. $A = s^2$; $s =$
6. $V = \frac{1}{3} Bh$; $B =$
7. $C = \pi d$; $d =$
8. $A = p + i$; $i =$
9. $A = \pi r^2$; $r =$
10. $A = bh$; $b =$
11. $V = lwh$; $h =$
12. $i = prt$; $p =$
13. $s = \frac{1}{2} gt^2$; $g =$
14. $A = \frac{1}{2} bh$; $h =$
15. $A = \pi ab$; $a =$
16. $V = Bh$; $h =$
17. $A = lw$; $l =$
18. $V = \pi r^2 h$; $h =$
19. $VP = k$; $P =$
20. $V = \frac{1}{3} \pi r^2 h$; $h =$
21. $l = ar^{n-1}$; $a =$
22. $V = \frac{1}{4} \pi d^2 h$; $h =$
23. $T = nc$; $n =$
24. $i = prt$; $r =$
25. $V = lwh$; $w =$
26. $p = 2l + 2w$; $l =$
27. $A = p + i$; $p =$
28. $C = 2\pi r$; $r =$
29. $p = 2l + 2w$; $w =$
30. $i = A - p$; $p =$
31. $A = p(1 + rt)$; $p =$
32. $f + s + t = 180$; $f =$
33. $s = \frac{1}{2} gt^2$; $t^2 =$
34. $P = s - (c + e)$; $s =$
35. $V = \frac{1}{3} \pi r^2 h$; $r =$
36. $V = lwh$; $l =$
37. $l = a + (n - 1)d$; $a =$
38. $s = \frac{1}{2} gt^2$; $t =$
39. $B = \frac{1}{12} lwt$; $w =$
40. $f + s + t = 180$; $s =$
41. $s = \frac{1}{2} n(a + l)$; $n =$
42. $T = 2\pi rh + 2\pi r^2$; $h =$
43. $l = a + (n - 1)d$; $d =$
44. $e = f + v - 2$; $v =$
45. $P = s - (c + e)$; $e =$

Science, industry, and commerce furnish innumerable examples of cases where it is necessary to derive one formula from another. For the most part these applications, however, involve situations that are far too technical for pupils of junior-high-school age, and care should therefore be taken to keep the cases selected for this work as simple as possible.

TEST NO. 43

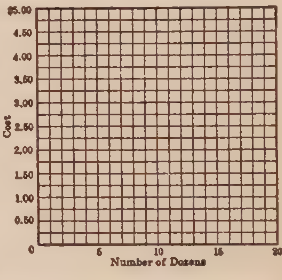
NAME.. DATE..................

CHECKED BY .... RIGHTS ....

GRAPHS OF FORMULAS

Time: 6 min.

1. On the accompanying section of squared paper draw a cost graph for bananas at 25¢ per dozen.



Complete each of the following statements:

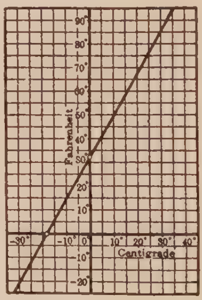
2. The graph in Ex. 1 shows that the cost of 5 doz. bananas is $..........

3. The graph in Ex. 1 shows that ..... doz. bananas can be bought for $3.25.

4. The graph in Ex. 1 shows that doubling the number of dozens .... the total cost.

5. The graph in Ex. 1 shows that the total cost of the bananas increases as the number purchased ....; that is, the cost to the purchaser depends upon the ....

6. This graph of the formula $F = 1.8\,C + 32$, which may be used in changing temperature readings on the centigrade scale to readings on the Fahrenheit scale and vice versa, shows that a reading of 10° C. (centigrade) corresponds to a reading of .... F. (Fahrenheit).



7. The graph in Ex. 6 shows that a reading of 32° F. corresponds to .... C.

8. The graph in Ex. 6 shows that a reading of $-25°$ C. corresponds to about .... F.

9. A Fahrenheit reading is always 1.8 times the corresponding centigrade reading, plus ....

10. Doubling a Fahrenheit reading does not double the corresponding .... reading.

After he has drawn the graph of a formula the pupil should see that he has done more than make a table of values and plot corresponding points. He should realize that what he has done is to draw an accurate picture of the relationship between the quantities which is expressed by the formula, and that from this picture he can frequently draw many interesting conclusions.

TEST NO. 41

NAME.. DATE....................

CHECKED BY .... RIGHTS ....

DEPENDENCE OF QUANTITIES

Time: 4 min.

In each of the blanks in the following statements insert the word which makes the best sense:

1. The cost of a sirloin steak depends upon the weight and the .... per pound.

2. The value of the algebraic expression $5x - 3$ depends upon the value of .....

3. The circumference of a circle depends upon the length of the .... or of the ....

4. The cost of sending a package by parcel post depends upon the .... of the .... and the distance to the place to which it is sent.

5. Doubling the length of the radius of a circle .... the circumference.

6. The number of yards of wall-paper border needed to go round a rectangular room depends upon the .... and .... of the room.

7. The number of theater tickets that can be bought with a 10-dollar bill depends upon the .... of each ticket.

8. The .... that a boy can walk in 3 hr. depends upon the number of miles that he can walk per ....

9. The volume of a cube depends upon the .... of an ....

10. The time that it takes me to fill in all the blanks on this page at an average rate of 5 blanks per minute depends upon the .... of ....

11. The interest received per year from an investment of $500 depends upon the .... of interest at which the investment is made.

12. The cost of excavating a rectangular cellar at a fixed price per cubic yard depends upon the ...., ..., and .... of the cellar.

13. Doubling the length of the radius of a circle multiplies the area by .....

When one quantity depends upon another for its value, the mathematician says that the first is a function of the second. At this stage the idea of functionality must, of course, be approached informally, but it is important that the pupil should see that this valuable mathematical principle is of everyday occurrence in ordinary life.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Contrast prognostic and diagnostic tests with reference to their nature and purpose.

2. What do you consider the best kind of prognostic test for determining which pupils are likely to profit by a study of mathematics beyond the ninth grade?

3. State clearly what you consider to be the advantages and disadvantages of the "essay-type" examinations in mathematics.

4. Consider the preceding topic with reference to the "new-type" examinations.

5. Criticize the present marking system in your school. Can you suggest a scheme that is better and at the same time feasible?

6. Discuss the main arguments that seem to you valid against the use of "extramural" examinations.

7. Mention any strong or weak points of standardized tests that are not given in this chapter and that you feel to be worth considering.

8. Discuss the place of tests in curriculum construction, stating the purpose and nature of such tests.

9. Select some important objective which you wish to attain in teaching mathematics and make a test relating to it that shall be both valid and easily scored.

10. Discuss the importance of tests in providing a proper basis for remedial instruction.

11. What is the difference between what is commonly called a survey test and one that is diagnostic? Does either one include the other?

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CHAPTER XII

HOMEMADE INSTRUMENTS

1. Educational Purpose

Historical Survey. The primitive mathematical instruments were naturally characterized by their simplicity. For example, the ruler or straightedge was designed for drawing a straight line, and the compasses for drawing a circle (probably the primitive instrument for this purpose was merely a string). For measuring an angle, a circle with a revolving radius was used very early. It was relatively late in the development of mathematics that instruments were so combined as to give us the modern transit.

Naturally, in the teaching of children we are attracted by the very simplicity of the early types and are led to see what the childhood of the race has to suggest for the use of the childhood of the individual. If the early instruments prove to be, through their freedom from complexity, more easily made and more readily understood than our later forms, we should take advantage of this fact and should act accordingly.

Chief Purpose. In considering the nature and use of homemade instruments the chief purpose to be kept in mind is to discover such simple types as shall be of the most help to the pupils in making their work interesting and in giving to it an appearance of greater reality. It is therefore proposed in this chapter to describe certain of these instruments both verbally and by illustrations, doing this in such a way as to encourage the pupils to make them and then to use them in their regular work.

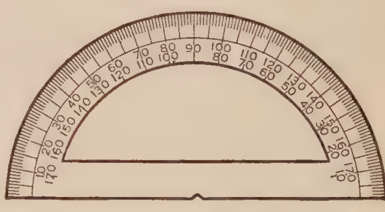
Naturally the list to be given is only suggestive of the general field. It is not expected that the teacher will feel obliged to duplicate all the instruments here mentioned. On the other hand, he will doubtless add to the list various types which he has found serviceable in his teaching.

2. Kinds of Instruments

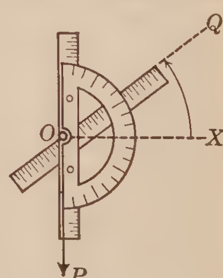
Instruments for Measuring Distances. Teachers will probably find it helpful if the suggested instruments are classified and described in some definite order, as is done in this and the following paragraphs.

It is needless to say that a great deal of profitable work in measuring can be done in the field without elaborate instruments of any kind. Rulers, yardsticks, and tape lines are always available or, if not, can easily be made by the pupils, and occasionally the older type of surveyors' chains or the more modern steel tapes can be procured. In any case, simple measuring devices of this kind can be had at no appreciable expense.

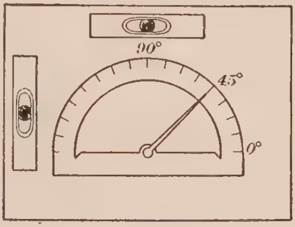
Instruments for Measuring Angles. It is possible to procure cardboard protractors at a stationer's at a slight cost, and even those of celluloid or brass are not expensive. It is not necessary, however, to buy such instruments, for the pupils can make them with a fair degree of accuracy. By doing this there is secured the double advantage of training the hand and eye and of making the protractor a complete circle. The important principles relating to the instrument can be taught more effectively with this form than with the conventional one here shown.



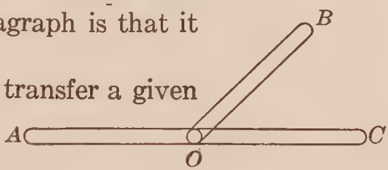
If a transit is not available, a simple device for measuring angles in the vertical plane can be made by using two rulers, a protractor, and a plumb line as here shown. The plumb line, which is attached to the protractor at O, is used to keep one ruler in position while sighting. By holding the instrument in this position vertical angles can be measured, horizontal angles being measured by holding the instrument flat.



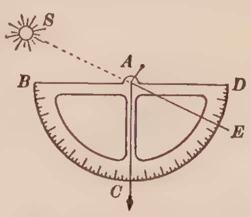
For measuring angles in the horizontal plane it is also interesting to have the pupil make what we may call a field protractor, a convenient instrument for solving various simple outdoor problems. For this purpose it is desirable to have a rather large protractor, such as is occasionally used for blackboard work. This should be fastened on a drawing board as here shown, a slender pointer being fixed so that it revolves like a moving radius. If two inexpensive carpenter's levels (homemade if necessary) are fastened to the board parallel to two adjacent sides and if the board be then placed on a simple tripod, the instrument is ready for use. The only advantage that this instrument has over the one described in the preceding paragraph is that it is more easily leveled.



If the pupil wishes merely to transfer a given angle from one position to another, the instrument here shown will be found convenient. It is easily made by pivoting one stick at the midpoint of another.

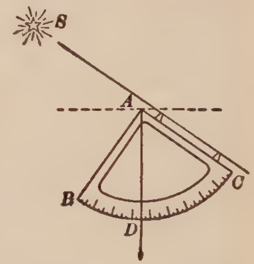


A convenient instrument for measuring the angle of elevation of the sun above the horizon may be made with a protractor and a plumb line as here shown. The plumb line is suspended from a long nail at A and should fall on the 90-degree mark. The shadow of the pin will then fall upon the protractor, as at E, and the angle of elevation is $\angle EAD$.



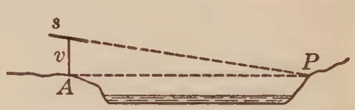
The pupil may also find the angle of elevation of the sun or of a star by means of a *quadrant*, — an instrument representing a quarter circle. It may be made by cutting a semicircular protractor in halves and attaching a plumb line at the center. When held in a vertical position as shown on page 362, with the points A and C in a line with the sun's rays, the angle of elevation of the sun is the same as $\angle BAD$.

To find the elevation of a star, the pupil should simply sight through C and A. It is possible that the teacher may wish to explain to the class that the angle of elevation of the north star gives the latitude of the observer, and in this way it is possible to find the approximate latitude of the place in which the pupil lives.



Instruments for Measuring Distances by Proportion. There are many historic instruments which make use of the properties of similar figures. They often show the interest of mathematicians of centuries ago in measuring, by their simple means, the heights of towers and hills, the width of rivers and lakes, the depths of wells and valleys, and, in general, any distances to inaccessible points. Some of the instruments were very crude, as are homemade ones today, but the mathematical principles underlying the instruments were generally as correct as those which we use in our own work.

One of the simplest methods of finding the approximate distance across a stream was to place a stick v vertically at the water's edge on one bank, with a shorter stick s that could be clamped to the first so as to be adjusted for sighting to a point P on the opposite bank. Leaving s clamped in this way to v, the observer then turned v about so that s pointed to some object on the level ground on the same side of the stream. Then the distance from A to that object would be the same as AP.



This method is illustrated in a book published in Venice in 1569 with a variant by which the observer used the visor of his cap in sighting across the stream. This curious old illustration is reproduced on the following page.

The pupils should understand that the above method illustrates the practical application of the second congruence theorem, — "Two triangles are congruent if two angles and the included side of one are respectively equal to two angles and

the included side of the other." Some of them may be able to find by means of a careful drawing the approximate error resulting from placing the rod at, say, an angle of 10° from the

Early Methods of Measuring Distances

From Belli's *Libro del Misvrar con la vista,* Venice, 1569, but representing essentially the method probably used by Thales

vertical but in the same plane as shown in the figure. A similar study might be made for the case in which the land at A sloped at an angle of, say, 20° from A to the object sighted.

The distance across a stream or a lake can also be found by the aid of a homemade transit. One pupil may hold a rod vertically at M and another, across the stream, may hold one vertically at N. The transit may be sighted horizontally and one boy may be told by the observer where to make the mark B, the other being told where to make the mark C. The observer may then sight to N, and the boy at M may be told where to place the mark K. Then by similar triangles,

$$\frac{AC}{AB} = \frac{CN}{BK},$$

whence

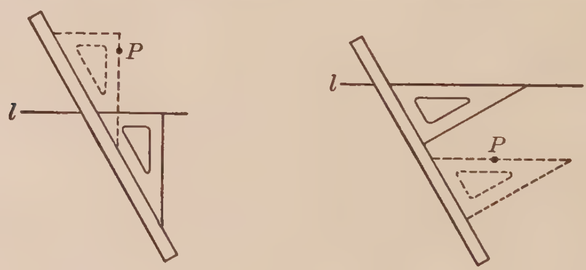
$$AC = \frac{AB \times CN}{BK}.$$

Knowing the length of AC, we find BC by subtracting AB.

A rough transit, made on the principle of the one used by surveyors, can be constructed if the interest of the pupils leads in that direction. Such a piece of work will create an appreciation of the instrument that could hardly be developed in any other way, on the part not merely of the makers but of the entire class as well. In a certain school one of the boys even went so far as to provide himself with an inexpensive telescope incased in a tin cylinder and mounted on a wooden tripod, protractors being used for the angle measurements. The use of this instrument gave to the whole class almost as good an appreciation of the first steps in trigonometry (as in the use of proportion) as they could have gained with a regular surveyor's transit.

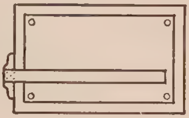
Instruments for Drawings and Constructions. Not all of the instruments that could be used by experts in drawing and construction work are needed in actual class work. We shall here mention several which, as stated in Chapter V, may be considered as luxuries rather than essentials. The essentials are, as already pointed out (page 60), the ruler (or straightedge) and the compasses, the uses of which need not be discussed further at this time.

In addition to these there is the draftsman's triangle, an instrument that is very useful in drawing perpendiculars and

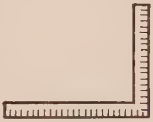


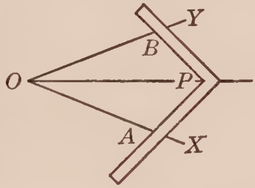
parallels. For example, the first of the following illustrations shows the method of drawing a perpendicular to l through the point P. The second shows the method of drawing a parallel to l through the point P. Such triangles are easily made by the pupils. When they make such instruments themselves they usually are more careful to preserve them.

The T-square is used by draftsmen for making long lines on paper pinned to a drawing board and for drawing parallels. The mathematical principle involved in the latter process is interesting to most pupils. The instrument, however, hardly ranks among homemade mathematical pieces.



The carpenter's "square" is rarely used in school, although it has numerous interesting applications and is easily cut out of cardboard for purposes of illustration. Its uses in drawing perpendiculars and parallels and in bisecting angles render it worthy of introduction into the work in intuitive geometry. Of these uses the most interesting is that of bisecting an angle. If the angle is $\angle XOY$, lay off OA equal to OB as shown in the figure. Then place the square so that A and B lie at equal distances from P, as measured along the square. By drawing OP the angle is bisected. The pupils should be asked to explain why this statement is true and should see that it depends upon the fact that the three sides of one triangle are respectively equal to the three sides of the other.





Instruments especially Helpful in Teaching. Among the devices which can be made by the pupils and which are especially helpful in teaching, the following are typical:

1. Long rulers for blackboard uses. These are often made in the school shop. A good size is 24 in. by $1\frac{1}{2}$ in. by $\frac{1}{4}$ in., and they are more convenient for use if they have handles.

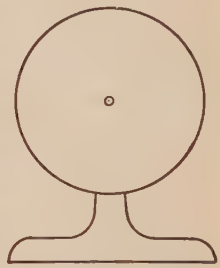
2. Wooden triangles shaped like draftsmen's triangles but made larger. They should have holes so that they can be hung upon nails placed at intervals underneath the chalk trays and thus be conveniently placed for use.

3. Wooden protractors, one or two being sufficient.

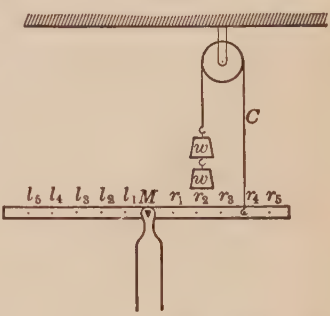
4. Cross-section board, a desirable feature for every classroom in mathematics, although hardly an instrument in the ordinary use of the word. A good foundation color is pea green, since it

shows the chalk marks easily from across the room. It will be found convenient to rule it in inch squares with a heavier line every five inches. Such a ruling into five spaces resembles the millimeter ruling frequently used in classes.

5. A circular blackboard. This is a great convenience, since it can turn a figure into various positions. In this way the pupil avoids the habit of thinking that a perpendicular to a line must always be a vertical line or that a triangle must always have a horizontal line for its base. Such a device consists simply of a circular board about a yard in diameter and so arranged that the base of the mounting fits into the chalk tray.



6. A device for illustrating the laws of directed numbers. This consists of a light bar balanced at M, as shown in the figure. Small screw hooks $r_1, r_2, r_3, \cdots$ and $l_1, l_2, l_3, \cdots$ are placed on the bar at equal distances to the right and left of M. Equal weights (w) are provided for the experiments which illustrate the laws.



The teacher should make three simple agreements with the pupils as follows:

a. Distances on the bar to the right of M are positive, distances to the left are negative.

b. Weights that pull up on the bar (down on the pulley) are positive, those that pull down are negative.

c. Counterclockwise rotation of the bar is considered positive, clockwise rotation is negative.

The following experiments will illustrate how the apparatus is used:

a. To find the product of $+2$ and -4 hang four weights on r_2. Since the bar turns clockwise, we see that $(+2)(-4) = -8$.

b. To find the product of -2 and -4 hang four weights on l_2. Since the bar turns counterclockwise, we see that $(-2)(-4) = +8$.

c. To find the product of $+2$ and $+4$ fasten one end of the string shown on r_2 and hang four weights on the other end which runs over the pulley. Since the bar turns counterclockwise, we see that $(+2)(+4) = +8$.

d. To find the product of -2 and $+4$ fasten one end of the string on l_2 and hang four weights on the other end of the string as was done in *c*. Since the bar turns clockwise, we see that $(-2)(+4) = -8$.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Make a list of all the necessary instruments that you would feel justified in asking for in your department in teaching junior-high-school mathematics.

2. How many of the instruments in the list given in No. 1 could, if necessary, be made by the pupils themselves?

3. Select a list of minimum essentials that you would wish to have included in the equipment of every classroom in mathematics in the junior high school.

4. What is there that is inherently valuable in asking pupils to make some of their own instruments for use in the study of mathematics? Illustrate your answer.

5. Have you any reason to believe that, generally speaking, girls are more skillful than boys in making effective instruments for class use?

6. Write a set of directions such as you would give to a painter for making the most convenient cross-ruled (squared) board for your classroom. Assume that pea-green paint is desirable and choose the most advantageous units.

7. Is the use of models in teaching geometry, for example, to be condemned or approved? Give the reasons for your answer.

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An instrument easily constructed by pupils.

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A few methods by which a straight line may be drawn.

CHAPTER XIII

MATHEMATICS CLUBS AND CONTESTS

1. Purpose and Organization of Clubs

Purpose. The purpose of mathematics clubs in high schools, normal schools, and colleges is naturally the same as that of the Pythagorean Brotherhood of ancient times, of which they are the remote descendants. This purpose was probably partly social and partly mathematical; it was probably intended that the society should foster an interest in mathematics and serve as a means of revealing the truths of the science to the world.

The purpose of our mathematics clubs of the present time may therefore be stated as being (1) to bring together kindred spirits, bound by an appreciation of the beauties and significance of mathematics; (2) to afford an opportunity for discussing the interesting features of the science; and (3) to inspire future teachers with the nobler phases of the subject so that they can in turn inspire their pupils.

Such clubs are found in many schools and colleges in the country, they usually meet once a month, they often publish their programs in the *American Mathematical Monthly* or *The Mathematics Teacher*, and they serve a very useful purpose.

Organization. These clubs are usually founded as the result of an informal talk by some of the teachers of mathematics, addressed to those who show an interest in the science. Election to the club is usually looked upon as a privilege, only those being considered who show some mathematical ability.

In general, the name of the organization is simply "The Mathematics Club of ———— High School," or college, or normal school, as the case may be. Some clubs adopt rather fanciful names, such as "Neo-Pythagoreans," "The Magic

Square," "The Mystic Hexagram," "The Pascal Triangle," "The Galois Group," or "The Cartesian Oval."

The organization is usually simple, the officers being a president, vice president, secretary, and treasurer. These officers sometimes have fanciful names, such as, respectively, "Transcendent Number, Surd, Imaginary, and Real," "Ordinate, Abscissa, Quadrant, and Origin," "Surd, Absurd, Rational, and Irrational," "Maximum, Minimum, Superior Limit, and Inferior Limit," or "Integral, Differential, Constant, and Variable." Such pleasantries often add a touch of humor to the routine of formal organization.

2. The Program

General Nature. The general nature of the program naturally varies with the school or college. An undergraduate club at Columbia, Harvard, or Chicago would necessarily have different topics from those of a club in a high school, technical school, or normal school. Since this work is written primarily for teachers in elementary and secondary schools or in teachers colleges or training schools, the subjects which might properly form a program have been selected with these institutions in mind.

In general, the topics most frequently selected for discussion refer to (1) the history of mathematics, (2) the curiosities of the science, (3) some elementary phase of algebra or geometry that can be discussed in such a way as to be interesting, or (4) some aspect of mathematics that connects the science with related fields. Several topics will be suggested as typical, and it is believed that these will suggest many others equally interesting.

Classification. For convenience in program-making the topics have been classified, not with a view to having any one program confined to a single field, but so that a committee can easily select a balanced intellectual ration for any particular evening.

Those relating to the history of mathematics can be supplemented very easily and extensively by consulting the "Topics for Discussion" which fill a page at the end of each chapter in Smith's *History of Mathematics* (Vols. I and II), and so they are, in the following list, purposely limited in number.

The names of mathematicians suitable for biographical sketches are typical of those concerning whom there is plenty of material available. The lives of men like Euclid and Diophantus, for example, are entirely unknown.

1. *History of mathematics.* The following are a few of the many interesting topics in the history of mathematics:

a. Definitions in arithmetic; in algebra; in geometry.

b. Symbols of operation.

c. Further symbols peculiar to algebra.

d. Greek and Roman numerals.

e. Our common (Hindu-Arabic) numerals.

f. Methods of adding and subtracting.

g. Methods of multiplication.

h. Methods of division.

i. The check of casting out nines.

j. Methods of finding roots of numbers.

k. Comparison of the modern analytic treatment of conics with the method of Apollonius.

l. Great periods in the development of algebra; of geometry; of trigonometry.

m. Such problems as the Josephus problem, the problem of the couriers, and work problems.

n. The development of the symbols of operation.

o. The development of the algebraic symbols.

p. History of the calendar, with a consideration of possible reforms.

q. Our common measures of length, area, and weight.

2. *Biographical sketches.* The following are some of the interesting names that might find place from time to time on the programs:

Pythagoras	Recorde	Mersenne	Monge
Plato	Tonstall	Descartes	Cauchy
Archimedes	Cardan	Pascal	Poncelet
Eratosthenes	Tartaglia	Wallis	Desargues
Hypatia	Vieta	Lagrange	Gauss
Boethius	Oughtred	Laplace	Cavalieri
Ptolemy	Harriot	Roberval	Lobachevsky

3. *Mathematical instruments.* The following are a few of the early instruments, some of which are still in use, together with the relatively late sextant and transit:

Quadrant	Sextant	Baculus	Speculum
Astrolabe	Transit	Plane table	Level

4. *Calculations.* The following topics relate to ancient and modern methods of calculating:

a. The abacus, various types
b. Lightning calculators
c. Modern calculating machines
d. The use of counters
e. Slide rules of various types
f. Logarithms of various types
g. Napier's rods
h. Finger computation

5. *Nature of mathematics.* The following topics will serve to reveal the general nature of mathematics in a way that is not usually feasible in class work:

a. What is mathematics?

b. Essential nature of algebra; its change from time to time as civilization developed.

c. The nature of elementary geometry compared with that of analytic geometry.

d. The nature and distinguishing features of modern projective geometry.

e. The fundamental concepts and assumptions upon which mathematics rests.

f. The special nature of the parallel postulate, and the effect upon geometry of denying its validity.

g. The nature and measurement of space.

h. A report upon Professor Keyser's *Philosophy of Mathematics*, or selected parts of the work.

i. The relation of mathematics to "the higher life."

6. *The nature of number.* For more advanced work in a club, the following topics on the nature of number, if simply treated from the historical standpoint, will be found very inspiring:

a. The so-called "natural numbers," — their nature and interesting features.

b. The development of "artificial numbers," — surds, fractions, negatives, transcendental, imaginary, complex.

c. Graphic representation of numbers of the above types.

d. Graphic representation of operations with numbers of the types mentioned above, including, for example, $(a+bi)^n$, or $r(\cos\theta + i\sin\theta)^n$.

e. The complex nth roots of unity, for $n = 3, 4, 5, 6$, and 7. The graphic solution of $x^5 - 1 = 0$.

f. The significance of $e^{\pi i} = -1$, and the developments which this equation suggests.

g. Scales of notation, with a discussion of the advantages and disadvantages of the scales of 2, 8, and 12.

7. *Elementary features of number theory.* The following topics can be so presented as to be within the range of interest of any mathematics club with pupils of grades nine to twelve:

a. History and nature of "perfect" numbers.

b. History and nature of "amicable" numbers.

c. Curiosities of numbers.

d. Magic squares, magic cubes, and magic circles.

8. *Amusing problems.* A never-failing source of interest is found in such topics as the following:

a. A report upon Professor Weeks's *Boys' Own Arithmetic*, especially upon some of its most interesting problems.

b. Logical fallacies.

c. Geometric fallacies.

d. Selected puzzles from books dealing with mathematical recreations of various types.

9. *Algebraic topics.* Algebra offers a wide range of interesting topics of which the following are types:

a. Various methods of factoring expressions of the type $ax^2 + bx + c$.

b. Showing that every quadratic equation can be solved by factoring.

c. Solving two simultaneous quadratic equations.

d. The solution of the general cubic equation.

10. *Geometric topics.* Demonstrative geometry offers to clubs in the senior high school an even wider range of topics than algebra, the following being types:

a. Various proofs of the theorem of Pythagoras.

b. Proofs of the angle sum of a polygon, with a consideration of various types, including cross polygons.

c. Critical points in a triangle, with a study of their positions as the triangle changes its form.

d. The nine-point circle.

e. The lunes of Hippocrates.

f. Geometry of the compasses.

g. Geometry of paper folding.

h. Geometric forms in nature.

i. The geometry of crystallography.

j. Various methods of trisecting an angle.

k. The duplication of a cube.

l. The squaring of a circle and the history of π.

m. Linkages.

n. The four tangents to two circles discussed as the circles change in size and position, and as they revolve and generate spheres. The relation of the problem to the study of eclipses.

11. *The nature of spaces.* Clubs always find the discussion of various spaces a topic of great interest. The work may be divided into small units as follows:

a. Lineland; life in a space of one dimension.

b. Flatland; life in a space of two dimensions; geometry in the same space, — some of its propositions and the methods of proof.

c. Four-space; life in a space of four dimensions; some idea of geometry in that space and of the methods of proof.

d. Elementary uses of mathematics in astronomy in a space of two dimensions; in our space of three dimensions.

e. Curvature of two-dimensional space; the result in the life of a Flatlander; the analogous case in a space of three dimensions.

f. Possibilities as to the nature of our space.

g. Time in relation to space.

3. Mathematics Contests

General Considerations. There is no reason why we should not have mathematics contests as well as public-speaking contests. They are equally as interesting and, for those who participate in them, they are quite as valuable. The fact that such contests are not held as often as debates in the English work is partly due to a lack of knowledge of the method of procedure. It is our purpose to give at this time a few of the more important details of the methods which should be used in carrying on a contest of this kind. The suggestions presented are based upon a contest that was recently given by the pupils in a certain junior high school. The contest was enjoyed greatly not only by the pupils and teachers of the school but by a large number of enthusiastic parents and outsiders as well.

Teams. There should be two teams, as is usual in an ordinary debate, together with suitable alternates for each. Experience has shown that five persons on each is a good working number. Each team should choose a captain to represent it whenever necessary.

Rules governing the Contest. To win the general contest a team must secure two out of three points. The contest should be divided into two parts, one written and the other oral. The decision should be based upon the results of the written and oral parts together with the general merit displayed. These bases for decision will now be discussed.

1. *Written contest.* The rules for the written part of the contest should be as follows:

a. A minimum course in mathematics upon which this part is based should be submitted to three competent judges, preferably people outside the school in which the contest is held. These judges should then be requested to formulate a series of questions and problems which shall constitute the basis for the written part of the general contest. The examination should be so constructed as to occupy an hour's time for each team and should be handed to the teacher in charge of the contest, in a sealed envelope, at the time when the event is to take place.

b. Both teams should be assembled at the same time and place, and the questions should then be given to each of the ten contestants. At the conclusion of the examination the papers should be given to the three judges for scoring as indicated below.

c. The papers should not be marked in the traditional way, but score cards should be used showing individual scores and team totals. The team with the greater total should be awarded one point in the general contest.

d. The result of the written part of the contest should not be announced until the general contest is concluded.

2. *Oral contest.* The rules for the oral part of the contest should be as follows:

a. The date and place should be set in advance, and the public should be admitted to the contest as in any ordinary debate.

b. The same judges who participated in the written part of the contest should judge the oral part.

c. The questions and problems for oral discussion should be prepared by the same judges as before, preferably in mimeographed or printed form, and should be handed to the teacher in charge when it is time for the contest to begin. There should be enough duplicate copies of each question to supply each contestant, each of the judges, and the teacher with a copy.

d. The teacher should then hand each member of both teams these questions one at a time, or in lieu of this should write them, one at a time, on the board and announce a definite number of minutes or seconds for silent study.

e. The teacher should tell each contestant in advance that whenever he is ready to recite he should stand. When time is called each captain should choose from those standing one member of the opposing team to answer the question or to put the solution of the problem on the board. The teams alternate thus in presenting their solutions.

f. Each team should be scored by the judges on a basis of 10 for each correct response. If the representative of either team makes a complete failure, his side should be given a score (penalty) of -2.

g. The team having the largest total score should be awarded one point in the general contest.

3. *General merit*. The rules for estimating the general merit of the two teams should be as follows:

a. There should be a timekeeper for the oral contest. He should record the total number volunteering for each solution from each team. Fifty special points should be awarded according to the ratio of the total number of volunteers on each team. For example, if the number of volunteers were 37 and 35 the teams would be scored respectively 25.7 and 24.3.

b. The judges should be requested to score both the written and oral expression of each team on a basis of 50 for perfection.

c. The team gaining the larger total number of points in *a* and *b* above should be awarded one point in the general contest.

The judges should then be asked to retire, and to return later with their decision, as is done in an ordinary scholastic debate.

Specimen Questions. Suppose that the contest is upon the geometry of position. The following are two specimen questions that might be given for each part, the total number being of course much larger:

1. *For the written contest.*

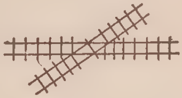
a. Tell how you would proceed to locate all points that are 1 ft. from a given point.

b. If you knew that some treasure was hidden equidistant from two given trees in a certain garden, tell how you would proceed to locate it.

2. *For the oral contest.*

a. Show by a drawing how you would proceed to locate some treasure that was known to be buried in a certain orchard 10 ft. from a given apple tree and equidistant from two other trees.

b. A railroad block signal is to be set at a point equidistant from two tracks which cross as here shown, and at a distance of 75 ft. from the "crossover." Draw a figure to show the point at which the signal is to be set. If there is more than one position, state what additional fact is needed to determine the precise position.



QUESTIONS AND TOPICS FOR DISCUSSION

1. State what you consider to be the purpose and the value of a mathematics club in the junior high school. State also the method of organization and the nature of such a club.

2. If the pupils should not see fit to organize a mathematics club, do you think it would be worth while for the teacher to do so? Give the reasons for your answer.

3. Plan a program which you may later use either in talking to a mathematics club or as a special assembly talk before the pupils of the entire school.

4. Discuss the social value of a mathematics club.

5. Suggest ways in which the plan of carrying on a mathematics contest as described on pages 375–377 may be improved.

6. Outline a list of topics in junior-high-school mathematics that the judges might later use as the basis for the written and oral parts of a general contest in mathematics.

7. Using the outline of No. 6, prepare a brief list of questions and problems that you would suggest if you were a judge in the written and oral parts of the general contest.

8. What seem to you to be the greatest values to be secured through mathematics contests in the junior high schools?

9. What, in general, should you say should be the nature of the material chosen for discussion in the meetings of a mathematics club?

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CHAPTER XIV

MATHEMATICAL RECREATIONS

1. THE SUBJECT IN GENERAL

Purpose. For one who likes mathematics the entire subject is a recreation, an eternal and endless riddle which he seeks to solve in minute portions. To this rule the mathematics of the elementary school is no exception when the subject is presented in the proper spirit. There is no reason why children, who take such a delight in simple puzzles, should be deprived of a similar pleasure when they study arithmetic or when, with increasing maturity, they begin their study of algebra and geometry. It is the purpose of this chapter to call attention to a few simple puzzles, many of which have some extended history and which have therefore brought pleasure to generations of learners of some of the mysteries of mathematics.

Literature. The literature of the subject is extensive. There are many books in various languages which present hundreds of problems of the recreational type, and the teacher who wishes to become more familiar with the subject should own one or more of these collections. In general they do not arrange the recreations with a view to their use in the schools nor with respect to their historical origin, and for this reason an effort has here been made to assist the teacher by some such classification.

The best results are obtained by giving one puzzle in a class period now and then as a kind of reward for good work.

2. NUMBER RECREATIONS

Elementary Number Puzzles. On page 382 is a brief list of very simple number puzzles, most of them derived from old books upon the subject, which have given pleasure to many thousands of boys and girls since they first appeared in print.

1. What is the number to which you add 1 when you take away 1?

Any such number, written in Roman numerals, as IV, IX, XIV, or XIX. Like most puzzles of this nature there is an element of unfairness in this because we do not take away 1 from the number; we simply take away a mark, I, from the representation of the number.

2. Write an even number using only figures representing odd numbers.

Numbers like $5\frac{3}{3}$, $7\frac{5}{5}$, or $11\frac{7}{7}$.

3. Write the number twelve thousand twelve hundred twelve.

The number appears as 13,212.

4. Write one hundred in our common numerals without using any zeros.

One way is $99\frac{3}{3}$.

5. Show that half of nine is four and that half of twelve is seven.

Cut IX and XII into two equal parts by a line parallel to the base and you have IV and VII.

Elementary Properties of Numbers. The following recreations involve no trick element; they represent real and interesting properties of numbers:

1. State at sight the sums of the following:

$5 + 6 + 7$	$21 + 22 + 23$	$221 + 222 + 223$
$9 + 10 + 11$	$39 + 40 + 41$	$329 + 330 + 331$
$19 + 20 + 21$	$110 + 111 + 112$	$499 + 500 + 501$

The solution consists in simply multiplying the middle number by 3. The results are 18, 30, 60, 66, 120, 333, 666, 990, 1500. The rule applies to any three consecutive numbers. To make it easy, make the middle number one that is readily multiplied by 3. For example, take 700; then the three numbers are 699, 700, 701.

2. State at sight the sums of the following:

$$3 + 4 + 5 + 6 + 7 \qquad 68 + 69 + 70 + 71 + 72.$$

The sums are 25 and 350. Simply multiply the middle number by 5. The rule applies to any five consecutive numbers.

3. State at sight each of these products:

$$\begin{array}{ccccccccc} 37 & 37 & 37 & 37 & 37 & 37 & 37 & 37 & 37 \\ \underline{3} & \underline{6} & \underline{9} & \underline{12} & \underline{15} & \underline{18} & \underline{21} & \underline{24} & \underline{27} \end{array}$$

The results are 111, 222, 333, 444, 555, 666, 777, 888, and 999. The reason is that $3 \times 37 = 111$, and therefore 6×37 must be twice as much, and so on. The trick appears in a French work by Ozanam, at least as early as the 1696 edition.

4. Find the value of

$$\begin{array}{rr} 1 \times 9 + 2 & 1234 \times 9 + 5 \\ 12 \times 9 + 3 & 12{,}345 \times 9 + 6 \\ 123 \times 9 + 4 & 123{,}456 \times 9 + 7 \end{array}$$

Then tell the value of $1{,}234{,}567 \times 9 + 8$
and $12{,}345{,}678 \times 9 + 9$

5. Find the value of $9 \times 12{,}345{,}679$, and then tell at sight the values of $18 \times 12{,}345{,}679$ and $63 \times 12{,}345{,}679$.

18 times the number is 2×9 times the number, or 222,222,222; and 63 times the number is 7×9 times the number, or 777,777,777. In the same way we can easily tell the product by 27 (3×9), 36, 45, 54, 72, and 81.

6. Tell at sight the sums of these numbers:

$$\begin{array}{r} 99{,}999 \\ \underline{99{,}999} \end{array} \qquad\qquad \begin{array}{r} 99{,}999 \\ 99{,}999 \\ \underline{99{,}999} \end{array}$$

In the first case we have $2 \times (100{,}000 - 1)$ or $200{,}000 - 2$, which is 199,998. Similarly, the second is $300{,}000 - 3$, or 299,997. The same trick is easily applied to 99,999 taken 4, 5, 6, 7, 8, or 9 times.

7. Ask a member of the class to write on the board any three numbers he wishes, of four figures each. Say that you will write underneath them three numbers and will at once tell the sum. Suppose that the pupil writes 7285, 5829, and 3456, as here shown. You then write 2714, 4170, 6543, and state that the sum is 29,997. How is it done?

$$\begin{array}{lr} \text{Written by pupil} & \left\{\begin{array}{r} 7285 \\ 5829 \\ 3456 \end{array}\right. \\ \text{Written by teacher} & \left\{\begin{array}{r} 2714 \\ 4170 \\ \underline{6543} \end{array}\right. \\ \text{Sum} & 29{,}997 \end{array}$$

You simply write the difference between 9999 and each number, which is done by taking each digit from 9. Thus: $9999 - 7285$ is 2714, your first number. The sum is then 3×9999, or $30{,}000 - 3$.

8. Take any three-figure number (say 348), reverse it (843), find the difference (495). If you will tell me the last figure, I will tell you the other two. How is this done?

The middle figure will be 9. The first figure will be 9 minus the last one.

9. Show that $45 - 45 = 45$.

The puzzle is a very old one. Write the two numbers as follows and subtract them as if they were large numbers, beginning at the right:

$$\begin{array}{r} 9+8+7+6+5+4+3+2+1=45 \\ \underline{1+2+3+4+5+6+7+8+9=45} \\ 8+6+4+1+9+7+5+3+2=45 \end{array}$$

10. If there are more than 366 pupils in your school, how can you be certain that at least two of them have the same birthday?

If 366 of them have different birthdays (including February 29 of a leap year), then the 367th one must duplicate one of these, for the entire list has been exhausted. If the 366 do not have different birthdays, then of course at least two must have the same one.

11. Why are we certain that there are at least two men in New York City with precisely the same number of hairs in their heads?

The case is similar to the preceding one. There are more than 3,000,000 men in the city and the maximum number of hairs for any one is only a few thousand. Even if the number of hairs were 1,000,000, then the 1,000,001st man would have the same number as one of the others.

Problems requiring Ingenuity in Connection with Numbers. Some of the following problems can be solved by certain rules, but most of them require simply ingenuity and patience.

1. In the statement $111 + 222 + 333 + 444 = 789$, cross out three figures in such a way as to have the addition correct.

Cross out 3 from the hundreds, 2 from the tens, and 1 from the units.

2. Write 100 using the same figure five times.

One way is $111 - 11$; another is $3 \times 33 + \frac{3}{3}$; and another is $5 \times 5 \times 5 - 5 \times 5$.

3. Write 100 using the nine digits, 1, 2, 3, $\cdots$, 9.

One way is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9$. Another is $96\frac{1752}{438}$. There are several others, one being $74 + 25 + \frac{3}{6} + \frac{9}{18}$.

4. Using all the ten figures, 0, 1, 2, $\cdots$, 9, without duplication, write two numbers whose sum is 100. One solution is $49\frac{38}{76} + 50$ and a certain fraction. Another is $78\frac{\ }{6} + 21\frac{45}{\ }$, where you are to complete the fractions.

The problem was first suggested about a century ago. It is too difficult for the pupils unless the solutions are given in part. The results are $49\frac{38}{76} + 50\frac{1}{2} = 100$, and $78\frac{3}{6} + 21\frac{45}{90} = 100$.

5. Write the number twenty using only four 9's.

This is comparatively simple. We have $9 + \frac{99}{9} = 20$.

6. Two persons play a game, the first taking any number from 1 to 10 inclusive; the second then adding to it any number from 1 to 10; the first then adding to this result any number from 1 to 10, and so on until 100 is reached, the one reaching 100 being the winner. Who is certain to win, and how does he do it?

The first can always win if he starts with 1. The reason is that the one who reaches 89 can certainly win, for the other must then name a number from 90 to 99, and so the first can reach 100. But the one who reaches 78 can certainly reach 89, and hence can reach 100. Similarly for the one who reaches 67, 56, 45, 34, 23, 12, and 1. Start with 1; whatever is added, add enough to make 12; then to make 23, and so on to 89 and 100.

7. In the division and multiplication here shown, write the figures in place of the letters that will make the work correct.

$$
\begin{array}{r}
ac \\
ac\overline{)abb} \\
\underline{ac} \\
cb \\
\underline{cb}
\end{array}
\qquad\qquad\qquad
\begin{array}{r}
icc \\
\underline{in} \\
ntt \\
\underline{icc} \\
iant
\end{array}
$$

The division is very simple; $a = 1$, $b = 4$, $c = 2$. In the multiplication, $i = 1$, $c = 4$, $n = 2$, $t = 8$, $a = 7$. Cases of this kind are easily made up by the class. It is a good scheme to take a key word of ten different letters. For example, in the multiplication case the key word is

i	n	o	c	u	l	a	t	e	s
1	2	3	4	5	6	7	8	9	0

Another good key word is "emulations."

8. What two-figure number multiplied by 8 makes 20?

The answer is 2.5.

9. A grocer arranged 32 boxes of sardines on a table, as in this square, remembering only that there were 9 on each side. A clerk sold 4, but the grocer did not notice it because there were still 9 on each side. He then sold 4 more, and still the grocer did not notice it because there were 9 on a side. He then sold 4 more, and still there were 9 on a side, and so the grocer thought he had the same number as before. How were the boxes arranged?

1	7	1
7		7
1	7	1

The problem is very old. It appears in a French work of 1612, and has often been printed since then. The arrangements are as follows:

2	5	2
5		5
2	5	2

28

3	3	3
3		3
3	3	3

24

4	1	4
1		1
4	1	4

20

10. What is the largest number that can be written with three 9's?

It is 9^{9^9}; that is, 9^{81}.

3. Problem Recreations

Trick Problems. Problems that derive their interest from a mere play on words are, of course, of little if any mathematical value. Nevertheless, they are sources of amusement and, as an occasional reward for mechanical drill, they have some value. The following are a few types:

1. On a limb of a tree sat 7 birds. A man shot 3 of them. How many remained?

None; the rest flew away.

2. Which should you prefer, an old $10 bill or a new one?

The old $10 is much better than the new "one" (dollar bill).

3. Explain this statement:

Ten fingers have I on each hand
Five and twenty on hands and feet.

Simply punctuate it correctly, thus:

Ten fingers have I; on each hand
Five; and twenty on hands and feet.

Of course, thumbs and toes count as fingers.

4. Do you know your A, B, C's? What is the middle letter?

The answer is B.

5. Which, if either, is the greater, six dozen dozen or a half a dozen dozen? How much greater?

Six dozen dozen is $6 \times 12 \times 12$; half a dozen dozen is 6 dozen or 6×12. The difference is 792.

6. Write fifty; add zero; add five; add one-fifth of eight. The sum has been called "the greatest thing in the world."

L + O + V + E, the E being one of the five letters in the word "eight."

7. Take 10 from 10 and leave 10.

When you take off a pair of gloves you take 10 fingers from 10 fingers and leave 10 fingers.

This puzzle appears in a German work of 1850.

8. Which is correct, 6 and 7 *are* 14, or 6 and 7 *is* 14?

Neither; but 6 and 7 *are* 13 and 6 and 7 *is* 13 are both correct.

9. Explain this doggerel verse from a book of the eighteenth century:

A thousand and one and fifty I'm clear,
Joined by fifty-one, will make right appear
Seven, one, four, two, eight, five, very plain
Whole numbers, and just five sevenths remain.

This is a specimen of hundreds of such absurdities of the eighteenth century. The first two lines give, in Roman numerals, MILLI. This is $\frac{5}{7}$ of the word MILLION, and $\frac{5}{7}$ of 1,000,000 is $714{,}285\frac{5}{7}$ as stated in the last two lines.

10. In a box are six apples. It is required to divide these among six boys, in such a way as to leave one apple in the box. How can this be done without cutting the apples?

Give one of the boys the box with the apple in it.

Ingeniously stated Problems. There are many problems, always impractical, which have an interest because of their ingenious statements. They are so worded as intentionally to

lead the reader on the wrong road, and hence they have some value in training the pupil to read with more care the problems which he is called upon to solve. The following are among the best-known types:

1. A snail crawling up a pole 10 ft. high climbs up 3 ft. each day and slips back 2 ft. each night. How long will it take to reach the top?

The problem is a very old one. At the beginning of the eighth day the snail is 3 ft. from the top. Therefore at the end of that day he reaches the top. A pupil is apt to say that the snail gains 1 ft. a day, and so it will take 10 da.

2. If 6 cats eat 6 rats in 6 min., how many cats will it take to eat 100 rats in 100 min., at the same rate?

Six, for 6 cats eat 1 rat in 1 min.

3. At one cut to a minute, how many minutes will it take to cut a strip of cloth 60 yd. long into strips 1 yd. long.

It will take 59 min., for only 59 cuts are necessary, the last two pieces being each 1 yd.

4. A bottle and a cork cost \$1.10, and the bottle costs \$1 more than the cork. How much did each cost?

The bottle cost \$1.05 and the cork 5¢.

5. A watermelon weighs $\frac{4}{5}$ of its weight and $\frac{4}{5}$ of a pound. How much does it weigh?

The answer is 4 lb. Since $\frac{1}{5}$ of its weight must be the $\frac{4}{5}$ lb., $\frac{5}{5}$ of its weight must be $5 \times \frac{4}{5}$ lb., or 4 lb.

6. If an apple balances with $\frac{3}{4}$ of an apple of the same weight and $\frac{3}{4}$ of an ounce, how much does the apple weigh?

The answer is 3 oz.

7. If a herring and a half costs a cent and a half, how much will a dozen and a half herrings cost?

The answer is a dozen and a half cents, or 18¢.

8. Which weighs the more, 1 lb. of gold or 1 lb. of feathers?

Feathers are weighed by avoirdupois weight and gold is weighed by troy weight. The avoirdupois pound is heavier than the troy pound. Therefore the feathers weigh the more. While it is not worth while to teach troy weight in our schools, the puzzle serves to call attention to its existence.

9. How many quarter-inch squares will it take to make an inch square, and how many quarter-inch cubes to make an inch cube?

The answers are sixteen and sixty-four.

10. Two fathers and two sons divided three apples among themselves, each receiving one apple. How could this happen?

There were three persons, a boy, his father, and his grandfather. This makes two fathers and two sons.

11. Three brothers divided four apples among themselves so that one had no more than the others, and yet no apple was divided. Explain how this was possible.

The first took two apples and each of the others took one. Then one had no more than the others, although the first had more than either of the others.

12. After cutting off 10% of a piece of cloth a merchant had 100 yd. left. How much had he at first?

The answer is not 110 yd., but $111\frac{1}{9}$ yd.

13. In a family party there were 1 grandfather, 2 fathers, 1 grandmother, 2 mothers, 4 children, 3 grandchildren, 1 brother, 2 sisters, 2 sons, 2 daughters, 2 married men, 2 married women, 1 father-in-law, 1 mother-in-law, and 1 daughter-in-law. How many were there in the party?

Seven, — an old man and his wife, their son and his wife and two daughters and one son.

14. Two Arabs sit down to eat, one with five loaves and the other with three, all the loaves having the same value. A third Arab then comes along and proposes to eat with them, promising to pay 8¢ for his part of the meal. If they eat equally and consume all the bread, how should the 8¢ be divided?

This problem, which is found in print as early as 1612, is one of the most deceiving of the simple recreations, most persons being tempted to say "5¢ and 3¢." Evidently, however, each Arab eats $2\frac{2}{3}$ loaves, so that the third Arab eats only $\frac{1}{3}$ of a loaf out of the three loaves, and $2\frac{1}{3}$ ($\frac{7}{3}$) loaves out of the 5 loaves. Hence he pays 1¢ to one and 7¢ to the other.

15. A man sold a farm for \$5000. He then bought it back for \$4500 and then sold it for \$5500. How much did he gain?

The answer is \$1000. He has spent farm + \$4500. He has received farm + \$5500. Hence he gained the difference, or \$1000.

16. A man having 5 pieces of chain of 3 links each asked a blacksmith how much he would charge to make them into one piece of chain. The blacksmith told him that the charge would be 10¢ to cut a link and 10¢ to weld a link. How much would it cost to do the work?

It would cost 60¢, for the work can be done by cutting and welding only three links. Cut the three links of one piece of chain. Weld these to unite the other four pieces.

17. An Arab left 17 camels to his three sons, the first to have half, the second to have a third, and the third to have a ninth. Not seeing how they could make the division, they appealed to a wise man, who simply lent them a camel, after which the first took $\frac{1}{2}$ of 18, or 9; the second took $\frac{1}{3}$ of 18, or 6; and the third took $\frac{1}{9}$ of 18, or 2. This made $9 + 6 + 2$ camels, or 17 camels, and so they returned the odd one to the wise man. Was the division correct?

This is also a very old puzzle. Of course the division is not correct. The first received $\frac{1}{2}$ of a camel too much, the second $\frac{1}{3}$ of a camel too much, and the third $\frac{1}{9}$ of a camel too much. The reason this could happen is that the father really disposed of only $\frac{17}{18}$ of his property, for $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$.

18. Two clerks start in an office at the same time, one at a salary of \$1000 for the first year and a raise of \$200 each year thereafter, and the other with a salary of \$1000 a year but with a raise of \$50 every half year. Which has the larger income?

The second clerk receives \$500 + \$550 the first year. For the first half of the second year he has \$600 and for the second half \$650, making \$600 + \$650 the second year, and so on. Therefore we have

Years	1	2	3
First clerk	\$1000	\$1200	\$1400
Second clerk	1050	1250	1450

That is, the second clerk receives the more.

19. Three women went to town, each with some apples for sale. The first had 27, the second 29, and the third 33, and each gave as many apples for 3 cents as the other. When they had sold all their apples they found that they had the same amounts. How could this happen?

At the first house they sold at the rate of 3 apples for 3¢, the first selling 21 apples, taking 21¢, having 6 apples left; the second selling 24, taking 24¢, having 5 apples left; and the third selling 30 apples, taking 30¢, and having

3 apples left. At the second house they sold the rest of their apples at 3¢ each. Hence

the first had 21¢ + 18¢, or 39¢;
the second had 24¢ + 15¢, or 39¢;
the third had 30¢ + 9¢, or 39¢.

20. In a certain town 3 % of the inhabitants are one-legged and half of the others go barefoot. How many shoes are necessary?

As many as there are inhabitants, and this is the case whatever per cent is taken. The one-legged people need 1 shoe each, and the rest average 1 shoe each.

21. Of the weight of a ham, 10 % is bone, 5 % is fat, and 1 % is rind. If you know the price per pound of the entire ham, what else must you know and what must you do to find the cost of the entire ham?

You must know the weight. The first sentence has nothing to do with the case.

22. A boy requested an increase in salary, whereupon his employer proceeded to show him that he was not really entitled to what he was getting, arguing as follows:

There are 365 da. in the year	365 da.
You sleep 8 hr. a day, or $\frac{1}{3}$ the time	122
This leaves only	243
You spend 8 hr. in rest and recreation	122
This leaves only	121
There are 52 Sundays	52
This leaves	69
You have $\frac{1}{2}$ da. off on Saturdays	26
This leaves	43
You have $1\frac{1}{2}$ hr. daily for lunch, 5 da. a week,	16
This leaves	27
You are allowed 3 wk. vacation	21
This leaves	6
There are 6 holidays, — New Year's Day, Washington's birthday, July 4, Columbus Day, Thanksgiving Day, and Christmas	6
So you really do not work at all.	0
Moreover, there are Memorial Day and Labor Day; so you owe me for those.	

Wherein is the fallacy?

23. A boy at camp goes to a spring to get some water. He has one can holding 3 qt. and another holding 5 qt. How can he bring back exactly 4 qt. of water?

He can fill the 5-quart can; pour out 3 qt. into the 3-quart can; throw away the 3 qt. in the 3-quart can, and then pour into it the 2 qt. that were left in the 5-quart can. Then he can refill the large can and from it fill the rest of the 3-quart can, this requiring 1 qt. There will then be left 4 qt. in the 5-quart can.

24. A showman, traveling with a wolf, a goat, and a head of cabbage, came to a stream where there was a boat that was large enough to allow him to transport only one of these at a time. If he left the wolf alone with the goat, the wolf would eat it; if he left the goat with the cabbage, the goat would eat it. How could he get them all safely across?

First take the goat across from bank A to bank B; then go back and get the wolf, leaving it at B and taking the goat back to A; then leave the goat at A and take the cabbage across; then come back to A and take the goat across.

Amusing Problems. The teacher in search of amusing problems for recreation purposes should arrange to have on the bookshelves of the school, available for pupils' use, one or more copies of *The Boys' Own Arithmetic*, by Raymond Weeks. The following four problems are here included as specimens of the interesting and very original collection.[1]

1. On the Manakin Road lives a Dog who barks for 40 minutes whenever someone goes by at night. What is the smallest number of passers-by that will keep him barking all night? Allow $10\frac{1}{2}$ hours.

2. If Mr. Thompson of Yell County, aged 51, and his Dog, aged 2 years and 4 months, can get 1 Rabbit in eight hours' hunting, how many could they catch in the month of November? December? January? February? Allow 12 hours for a day's work, and deduct 4 days in each month for Sundays.

3. A Boy and his Sister, ages 7 and 8, are eating Watermelons. Working jointly, it takes them 5 minutes to consume the first Melon, 10 minutes the second, 15 minutes the third, etc., etc.

[1] By permission from *The Boys' Own Arithmetic*, by Raymond Weeks. Copyright by E. P. Dutton & Company.

What will be the number of the Melon which will require 1 hour and 35 minutes for its consumption?

4. Cousin Larry, aged 32, tells us of having a Tooth pulled. He says that, as his jaw lay on the back of the chair, it (the jaw) was 4 feet and 8 inches below a large, heavy, solid, bronze Chandelier, which hung directly above it (the jaw). He says that the Dentist, after 1 or 2 preliminary horizontal tugs, took hold with the forceps (a word derived from the Latin *formus*, meaning warm, and *capio*, I take; that is, I take while warm or hot: always seize every chance to acquire useful information), telling him that it would not hurt. He says that the Dentist then gave 7 vertical tugs, each of which raised him (Cousin Larry) eight inches, and that the Tooth came out when he (Cousin Larry) struck the Chandelier. He says that the distance from the point of impact with the Chandelier to the ceiling was 6 feet and 10 inches. How many tugs would it have required if there had been no Chandelier? Use imagination.

Problems having Some Relation to Geography. The following are types of puzzle problems relating to geography:

1. Assuming that the date line is a meridian, how many hours elapse from the time when today begins to appear on the earth to the instant when it finally disappears?

Forty-eight. It takes 24 hr. to cover the earth and 24 hr. to get off.

2. A hunter was 1500 yd. east of a polar bear and had a gun that carried only 1000 yd., and yet he shot and killed the bear. How was it possible?

He was near the north pole where the distance round the earth on a small circle was 2000 yd. He was therefore only 500 yd. west of the bear, so he easily shot it.

4. Algebraic Recreations

Problems involving Elementary Algebraic Symbolism. Many problems which were formerly classified as difficult are now readily solved by the use of simple algebraic symbolism. The following are a few types:

1. Think of any four numbers of one figure each. Double the first and add 1; multiply the result by 5 and add the second; double the result and add 1; multiply the result by 5 and add

the third; double and add 1; multiply by 5 and add the fourth. Tell me the result and I will tell you the four numbers.

Simply subtract 555 from the result, and the remainder will have for its digits the numbers required.

This is one of the many problems that can easily be explained by the first steps in algebra. It is, however, better than most others of the kind.

2. Why is it that if we add 1 to the product of two numbers which differ by 2, the result is the square of the number between them? For example, take 11 and 13; then we readily see that $11 \times 13 + 1 = 143 + 1 = 144$, which is the square of 12.

For if the numbers are n and $n + 2$, their product, plus 1, is $n^2 + 2n + 1$, which is the square of $n + 1$.

3. Tell a person to think of a number, to square it, to square the next larger number, and then tell you the difference between the squares and you will tell him the number. How is it done?

We have n and $n + 1$; then the squares are n^2 and $n^2 + 2n + 1$. The difference is $2n + 1$. Hence, subtract 1 and divide by 2 and you have the number selected.

4. Tell a person to think of two numbers, each less than 9. Then add 1 to twice the first and multiply the result by 5, and to this product add the second. Then subtract 5, ask for the result, and state the two numbers.

These will be the digits of the result. The reason is seen by considering the following: Let x and y be the numbers. Then we have $2x + 1$, $10x + 5$, $10x + 5 + y$, $10x + y$, a number whose digits are x and y.

5. Take any number of three digits in which the difference between the first and third digits exceeds 1. Form a new number by reversing the order of the digits. Find the difference (d) between these two numbers. Form another number (n) by reversing the order of the digits of d. Then add n and d, and whatever number you start with, I can tell you the result. Explain the method.

The result is always 1089. The process is as follows:

$$
\begin{array}{llll}
 & & 100a + 10b + c & \\
 & & \underline{100c + 10b + a} & \\
 & 100(a - c) & \qquad\qquad + (c - a) & \\
\text{or} & 100(a - c - 1) & + 90 + (10 + c - a) & = d \\
 & \underline{100(-a + c + 10)} & \underline{+ 90 + (-1 - c + a)} & \underline{= n} \\
\text{Adding,} & & 900 + 180 + 9 = 1089 &
\end{array}
$$

6. Show how to find a number which when divided by 5 or 7 has a remainder 1, but which is exactly divisible by 3.

This is one of the favorite indeterminate cases of the early writers. It appears as early as 1628. Evidently the number is of the form $5n+1$ and also of the form $7m+1$, since there is a remainder 1 when divided by 5 or 7. It must also be of the form $3p$. Hence it is of the form $5 \times 7k+1$, or $35k+1$, in which k must be such that $35k+1$ is divisible by 3, or that $33k+2k+1$ is divisible by 3. Evidently this is the case if $k=1, 4, 7, 10, \cdots$. Hence the required number, $35k+1$, is 36, 141, 246, 351, $\cdots$.

7. To find the difference in age between yourself and another who is older than you, first subtract each digit of your age from 9; then tell the other person to add the number thus formed to his age; this result will exceed 100; tell him to cancel the left-hand figure and add it to the right-hand figure. The result is the difference in age. Give the reason.

If your age is a and the other's is b, the successive operations are as follows: $99-a$, $b+99-a$, $b+99-a-100+1=b-a$.

8. Tell a member of the class to select two numbers and to tell you their quotient and their difference and you will tell him the numbers. How is this done?

The smaller number is equal to the difference divided by the quotient decreased by 1. If $\frac{a}{b}=q$ and $a-b=d$, then it is easily shown by algebra that $b=\frac{d}{q-1}$.

9. Find two fractions such that their difference is equal to their product. One such case is $\frac{1}{2}$ and $\frac{1}{3}$. Find two others.

If the fractions are x and y, we have $x-y=xy$. Now let $x=\frac{1}{2}$; then $y=\frac{x}{x+1}=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$, as stated. But we may let x equal any other fraction and find the corresponding value of y. There is, therefore, an infinite number of results. It is easy to graph the equation.

The problem was known as early as the time when America was discovered and has interested thousands of pupils ever since.

10. If two fractions are equal and have equal numerators, of course the denominators must be equal. Since $\frac{0}{2}=0$, and $\frac{0}{3}=0$, we have $\frac{0}{2}=\frac{0}{3}$, and hence $2=3$.

The fallacy is in the first sentence. The statement is true only if the numerators are not zero.

11. If $\frac{a}{b} = \frac{c}{d}$, and if $a > b$, then c must be greater than d. But $\frac{1}{-1} = -1$, and $\frac{-1}{1} = -1$, and so $\frac{1}{-1} = \frac{-1}{1}$. But $1 > -1$, and hence -1 must be greater than $+1$.

The fallacy lies in the use of "greater." The word should be changed to "numerically greater."

12. Take any two whole numbers which differ by 1 and tell me their product and I will tell you the numbers.

They will be square roots of the next larger and next smaller squares. For example, if the product is 132, the next larger square is 144 and the next smaller is 121. Then the numbers are 12 and 11.

Algebraically, the numbers may be represented by n and $n + 1$. Their product is $n^2 + n$. The next larger square is $n^2 + 2n + 1$, and the next smaller square is n^2. The square roots of these are $n + 1$ and n, the numbers.

13. Explain this verse from an old book of the 18th century:

If ten by ten divided be,
And ten added, the sum is three.

Algebraically we have $\frac{x}{x} + x = \frac{x + xx}{x}$. But in Roman numerals, $x = 10$. Hence we have $\frac{10 + 20}{10} = 3$.

Recreations based upon Equations. There are a large number of recreations involving fallacies in the solution of equations and involving problems which derive their interest from the way in which they are stated. A few of these are as follows:

1. All numbers are equal, as in the case of 5 and 7. This can be shown as follows: Suppose that $4a = 6b$, which is certainly true if $a = 3$ and $b = 2$, or if $a = \frac{3}{2}$ and $b = 1$, and for various other values.

Then $\quad 14a - 10a = 21b - 15b,$

or $\quad 15b - 10a = 21b - 14a,$

whence $\quad 5(3b - 2a) = 7(3b - 2a).$

Dividing by $3b - 2a$, we have

$$5 = 7.$$

This is typical of an unlimited number of such fallacies. They all depend upon the error of trying to divide by zero. The pupil does not ordinarily notice that $3b - 2a$ is zero, because we took $4a$ equal to $6b$, or $2a$ equal

to $3b$, so that $3b - 2a = 0$. Division by zero is impossible. To take a simpler case, $2 \times 0 = 3 \times 0$; if we could divide by 0, we should have, by canceling, $2 = 3$.

2. Avoiding the fallacy of No. 1, we can still prove that, for example, $9 = 5$, as follows:

Since $$9 + 5 = 2 \times 7,$$
multiplying by $9 - 5$, $$9^2 - 5^2 = 2 \times 7 \times 9 - 2 \times 7 \times 5,$$
whence $$9^2 - 2 \times 9 \times 7 = 5^2 - 2 \times 7 \times 5.$$

Adding 7^2, we have
$$9^2 - 2 \times 9 \times 7 + 7^2 = 5^2 - 2 \times 5 \times 7 + 7^2,$$
or $$(9 - 7)^2 = (5 - 7)^2,$$
whence $$9 - 7 = 5 - 7,$$
and, adding 7, $$9 = 5.$$

The fallacy lies in the fact that because $(9 - 7)^2 = (5 - 7)^2$, it does not follow that $9 - 7 = 5 - 7$. For example, because $2^2 = (-2)^2$, it does not follow that $2 = -2$. The proper statement is
$$9 - 7 = \pm (5 - 7),$$
from which we must select the correct numerical statement, namely,
$$9 - 7 = -(5 - 7) = 7 - 5.$$

No. 1 showed the first type of fallacy to be avoided; No. 2 shows the second type.

3. All numbers are equal. For let $a > b$ and let $a - b = c$; then $$a - b = c,$$
whence $$a = b + c.$$

Multiplying by $a - b$, we have
$$a^2 - ab = ab + ac - b^2 - bc,$$
or $$a^2 - ab - ac = ab - b^2 - bc,$$
whence $$a(a - b - c) = b(a - b - c).$$

Dividing by $a - b - c$, we have
$$a = b.$$

Hence $a = b$ even if $a > b$.

The fallacy is the same as in No. 1: because $a - b = c$, $a - b - c$ must be zero and cannot be used as a divisor.

4. All numbers are equal. Let the sum of any two numbers, a and b, be equal to $2d$. Then

$$b = 2d - a,$$

and $$2d - b = a.$$

Multiplying, $$2db - b^2 = 2da - a^2.$$

Subtracting from $$d^2 = d^2$$

we have $$d^2 - 2db + b^2 = d^2 - 2da + a^2,$$

or $$(d - b)^2 = (d - a)^2,$$

whence $$d - b = d - a,$$

and hence $$b = a.$$

5. We know that

1 inch square = 1 square inch.

Multiplying equals by 2, we have

2 inches square = 2 square inches,

which is not true. Hence the axiom is false.

"One inch square" is not a numerical quantity; it is the shape of a figure. Twice 1 inch square is not 2 inches square. A similar fallacy is seen in No. 6.

6. Since a glass half full equals a glass half empty, multiplying both sides of the equation by 2, we have

A glass full = a glass empty.

7. The multiplication axiom is false, because

2 ft. = 24 in.

and 3 ft. = 36 in.

Multiplying member by member,

6 ft. = 864 in.

which is false.

Through such fallacies as Nos. 1–7 pupils will learn the necessity of precision of thought and of statement better than through a mere scientific discussion of the validity of axioms.

8. Given the equation $12x - 28 = 9x - 21$, we have

$$4(3x - 7) = 3(3x - 7).$$

Dividing by $3x - 7$, we see that

$$4 = 3.$$

The pupil should be prepared to infer that the fallacy is that of No. 1 or No. 2. He should see that he has probably used zero as a divisor. This is the case, because $x = \frac{7}{3}$ in the equation, whence $3x = 7$, and so $3x - 7 = 0$.

9. A man is twice as old as his wife was when he was as old as she is now. When she is as old as he is now the sum of their ages will be 100 yr. Find their ages now.

As in most puzzles of this kind, the difficulty lies chiefly in the phraseology. The problem involves the two equations

$$m = 2(w - \overline{m - w})$$

and $$m + m - w + w + m - w = 100.$$

Solving, $$m = 44\tfrac{4}{9}, \qquad w = 33\tfrac{1}{3}.$$

10. Mary is 18 yr. old. Ann is twice as old as Mary was when Ann was as old as Mary is now. How old is Ann?

Such puzzles, as also No. 9, are merely verbal. The mathematics is simple. Let us say that x years ago Ann was as old as Mary is now; that is, she was then 18 yr. old.

Then $$a - x = 18,$$

and also $$a = 2(18 - x).$$

Hence $$a = 18 + x,$$

and so $$18 + x = 2(18 - x).$$

Solving, $$x = 6.$$

Hence $$a = 2(18 - 6) = 24.$$

11. A poor man about to cross a bridge was asked by a gypsy if he wished to be rich. When the man naturally replied that he did, the gypsy said, "Then I will give you a charm of wonderful power; it will double your money every time you cross the bridge. You may cross as many times as you wish provided you will pay me 40¢ after each crossing." The poor man saw untold wealth, and crossed the bridge, finding to his delight that his money was doubled. He then paid the gypsy 40¢ and crossed again. The wonderful thing again happened, and again he paid the 40¢. He then crossed a third time, his money doubling as before, but when he paid the gypsy 40¢ he found that he had nothing left, and he went away poorer than when he began. How could this have happened?

As usual, the difficulty comes from too many words. If he started with x cents, we have successively x, $2x - 40$, $4x - 80 - 40$, and $8x - 160 - 80 - 40$. But this last is zero. Hence, solving, $x = 35$. So he started with 35¢. It will be interesting to ask the class how it would have turned out if he had started with 40¢; with \$1. It is possible to suggest an economic lesson in business, of not starting with too little capital.

12. Given $x + 2 = y$ and $x = y - 3$, if we substitute the value of x in the first equation, we have $y - 3 + 2 = y$, whence $-3 + 2 = 0$, and $2 = 3$. Where is the fallacy?

The equations are inconsistent. If $x + 2 = y$, then $x = y - 2$. If this is so, the second equation cannot be true.

13. Solve the equation $3 + \dfrac{1}{x-2} = \dfrac{3x-5}{8+x}$. Since the first member reduces to

$$\frac{3x-5}{x-2},$$

we have
$$\frac{3x-5}{x-2} = \frac{3x-5}{8+x},$$

whence the denominators must be equal, and

$$x - 2 = 8 + x,$$

and
$$-2 = 8.$$

Wherein lies the difficulty?

The root is $x = \frac{5}{3}$. Hence $3x - 5 = 0$, and we have assumed that because

$$\frac{0}{-2} = \frac{0}{8+}$$

the denominators must be equal, which is not true, — a case already discussed.

14. A man having lost a purse full of coins said that he did not know how many coins he had, but that when he counted them by 2's, by 3's, or by 5's there was always 1 over, and when he counted them by 7's there were no coins left over. How many coins were there?

This is a favorite type of indeterminate problems. This particular one is found in books of the seventeenth century.

The number must be of the form $30x + 1$ in order to meet the first three conditions. Since this must be divisible by 7, it must be of the form $7y$.

Hence
$$30x + 1 = 7y.$$

If $x = 1$ or 2, y is a fraction; but if $x = 3$, then $y = 13$ and the number is 91. Other values may be found by letting $x = 10, 17, 24, \cdots$, increasing each time by 7.

There are numerous other recreations of the type just given that the teacher can utilize if he wishes. These can be found by consulting some of the sources given in the bibliography.

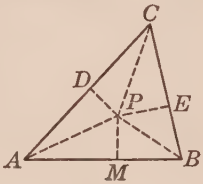
5. Geometric Recreations

Faulty Construction. The following fallacies show the danger of faulty construction of figures:

1. Every triangle is isosceles.

Let ABC be any $\triangle$ in which AC is not equal to BC.

Bisect $\angle C$ and construct the $\perp$ bisector of AB, letting it meet the bisector of $\angle C$ at P. They must meet, for if they were $\parallel$, the bisector of $\angle C$ would be $\perp$ to AB and hence would bisect it, thus coinciding with the $\perp$ bisector MP. This would be possible only if $AC = BC$, which is contrary to what is assumed above.



Draw $PD \perp$ to AC and $PE \perp$ to BC.

Then, since CP bisects $\angle C$, we have $PD = PE$; and since MP is the $\perp$ bisector of AB, then $AP = BP$.

Then $\triangle\ APD$ is congruent to $\triangle\ BPE$, and hence $AD = BE$.

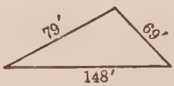
Similarly, $\triangle\ PDC$ is congruent to $\triangle\ PEC$, and hence $DC = EC$.

Adding, $AD + DC = BE + EC$, or $AC = BC$.

Hence every $\triangle$ is isosceles.

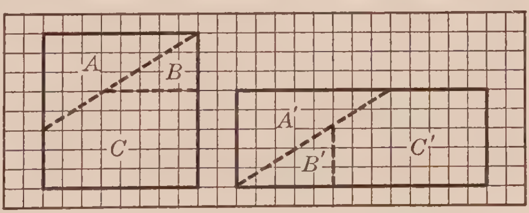
2. Find the area of this triangle to the nearest 0.1 sq. ft.

You may use the formula for the area of a $\triangle$ in terms of the three sides, or if you prefer, draw the figure to scale, measure the altitude of the $\triangle$, and then use the formula $A = \frac{1}{2}\,bh$.



3. Any number is equal to zero.

On a piece of squared paper mark out a square which shall be 8 by 8, and then draw lines dividing it into three parts A, B, C, as shown below.



Then mark out a ▭ which shall be 5 by 13, and divide it into three parts such that $A' = A$, $B' = B$, and $C' = C$, as shown.

The number of small squares in the large square is 8×8, or 64, and the number of small squares in the ▭ is 5×13, or 65.

Hence $$65 = 64,$$

or $$1 = 0.$$

Multiplying these equals by any number, say 25, we have

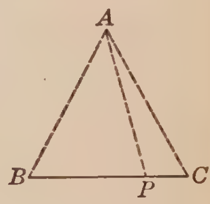
$$25 = 0,$$

and hence any number is equal to zero.

Incorrect Statement of Proposition. The following fallacy illustrates the frequent tendency to misquote a proposition:

Any point on a line bisects it.

In the figure below let BC be any line and P any point on it. Construct an isosceles $\triangle ABC$ upon BC as base, and draw AP.



Since $\angle B = \angle C$,

$AB = AC$,

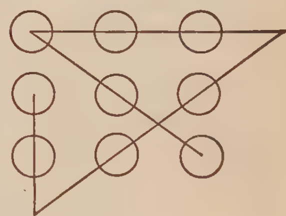
and $AP = AP$,

then $\triangle ABP$ is congruent to $\triangle ACP$.

$\therefore BP = PC$,

or any point on a line bisects it.

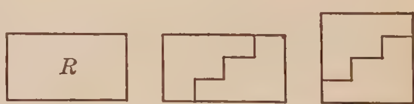
Problems in Drawing and Paper Cutting. The following are illustrative of the interesting puzzles that can be given with respect to the drawing or cutting of geometric figures, or to optical illusions:



1. Without lifting the pencil from the paper, draw four straight lines (that is, a broken line of four segments) passing through the centers of the nine circles here shown.

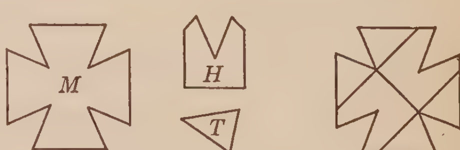
The required lines passing through the nine centers are here drawn, but when given to the class the circles only should, of course, be shown.

2. Required to cut the rectangle R into two equal parts which, placed side by side, will form a square.



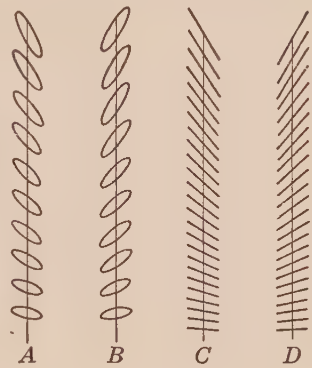
The solution is given by the two figures shown above at the right.

3. Required to put four triangles like T and two seven-sided figures (heptagons) like H together so as to form the Maltese cross M.



The solution is given in the figure at the right. Many similar problems are easily constructed by paper cutting.

4. By simply looking at the lines A, B, C, D make up your mind which, if any, are parallel. Write the statement and then, with a pair of dividers or a piece of paper, determine which, if any, of your answers are correct.



There are, of course, many other optical illusions of a similar nature that can be easily given. The chief value of such exercises lies in the use that can be made of them in leading pupils to see the need for proving statements in geometry. Although important conclusions may be deduced from a careful examination of a figure, such cases show that observation unsupported by some reliable check is worthless as a means of proving the truth of geometric statements.

QUESTIONS AND TOPICS FOR DISCUSSION

1. Write a short paper stating the value of mathematical recreations in the junior high school.

2. Where in the curriculum do you consider the best place for mathematical recreations? Give the reasons for your answer.

3. Write an outline of a program of interesting mathematical recreations, not presented in this chapter, that you consider appropriate for each of the three grades of the junior high school.

4. What use can be made of optical illusions other than to furnish amusement for the pupils?

5. Discuss the question as to whether mathematical recreations make different appeals to the different types of ability found among pupils in the same grade, or whether they make an appeal that is universal.

6. Show how the pupils' interest in the study of algebra may be increased through an appreciation of certain mathematical recreations.

7. What are some of the mistakes likely to be made, and how may they be avoided, in teaching recreational material in mathematics?

8. Read and be able to report in class upon one of the references in the bibliography given at the end of this chapter.

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